B. Sc. Part II   Paper IV

Group B.

$\boxed{\text{Vector Calculus}}$

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SCALAR TRIPLE PRODUCT

* $\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$ is a scalar quantity. Hence its name.

* $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{b}\cdot(\vec{c}\times\vec{a})$
$= \vec{c}\cdot(\vec{a}\times\vec{b}) = (\vec{b}\times\vec{c})\cdot\vec{a}$
$= (\vec{c}\times\vec{a})\cdot\vec{b}$

i.e. dot and cross can be interchanged.

Also, the position of $\vec{a}, \vec{b}, \vec{c}$ can be changed as $\vec{a}$ to $\vec{b}$, $\vec{b}$ to $\vec{c}$, $\vec{c}$ to $\vec{a}$.

* $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$.

* If two vectors are identical, then the scalar triple product $= 0$
i.e. $[\vec{a}\ \vec{a}\ \vec{b}] = 0 = [\vec{a}\ \vec{b}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{b}]$

* $[\vec{i}\ \vec{j}\ \vec{k}] = 1$

* $[\vec{a}\ \vec{a}\ \vec{a}] = 0$

* If the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

* If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Examples

I. Find $(2\vec{i}+3\vec{j}-5\vec{k})\times(3\vec{i}-2\vec{j}+6\vec{k}).(5\vec{i}+\vec{j}+\vec{k})$.

Solution

The above expression $= \begin{vmatrix} 2 & 3 & -5 \\ 3 & -2 & 6 \\ 5 & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 5 & 1 & 1 \\ 3 & -2 & 6 \\ 5 & 1 & 1 \end{vmatrix} (R_1 \to R_1 + R_2)$$

$= 0.$

Alternatively

$(2\vec{i}+3\vec{j}-5\vec{k})\times(3\vec{i}-2\vec{j}+6\vec{k})$

$= 9\vec{j}\times\vec{i} - 15\vec{k}\times\vec{i} - 4\vec{i}\times\vec{j} + 10\vec{k}\times\vec{j} + 12\vec{i}\times\vec{k} + 18\vec{j}\times\vec{k}$

$= -9\vec{k} - 15\vec{j} - 4\vec{k} - 10\vec{i} - 12\vec{j} + 18\vec{i}$

$= 8\vec{i} - 27\vec{j} - 13\vec{k}$

~~Also~~, $\therefore (2\vec{i}+3\vec{j}-5\vec{k})\times(3\vec{i}-2\vec{j}+6\vec{k}).(5\vec{i}+\vec{j}+\vec{k})$

$= (8\vec{i} - 27\vec{j} - 13\vec{k}).(5\vec{i}+\vec{j}+\vec{k})$

$= 8\times5 - 27\times1 - 13\times1$

$= 40 - 27 - 13 = \underline{\underline{0}}.$

i: Find $\cancel{\vec{a}\times\vec{c}}\ (\vec{a}\times\vec{b})\cdot\vec{c}$ where
$\vec{a} = 2\vec{i} - 3\vec{j} - 1\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$,
$\vec{c} = \vec{i} - \vec{j} + 2\vec{k}$.

Soln

$$\vec{a}\times\vec{b} = (2\vec{i} - 3\vec{j} - 1\vec{k})\times(2\vec{i} + \vec{j} - \vec{k})$$

$$= -6\vec{j}\times\vec{i} - 2\vec{k}\times\vec{i} + 2\vec{i}\times\vec{j} - \vec{k}\times\vec{j} - 2\vec{i}\times\vec{k} + 3\vec{j}\times\vec{k}$$

$$= +6\vec{k} - \cancel{2\vec{j}} + 2\vec{k} + \vec{i} + \cancel{2\vec{j}} + 3\vec{i}$$

$$= \cancel{3\vec{i}}\ 4\vec{i} + 8\vec{k}.$$

$\therefore (\vec{a}\times\vec{b}) = 4\vec{i} + 8\vec{k}$

Now, $(\vec{a}\times\vec{b})\cdot\vec{c} = (4\vec{i} + 8\vec{k})\cdot(\vec{i} - \vec{j} + 2\vec{k})$

$$= 4\times 1 - 0 + 8\times 2$$

$$= 20.$$

Alternatively

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} 2 & -3 & -1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} \quad R_1 \to R_1 - R_2$$

$$= \begin{vmatrix} 0 & -4 & 0 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4(2\times 2 + 1\times 1)$$

$$= 4\times 5 = \underline{20}$$

Impt. Volume of a parallelepiped [whose edges are $\vec{a}, \vec{b}, \vec{c}$] $= \vec{a}\cdot(\vec{b}\times\vec{c})$

$$= [\vec{a}\ \vec{b}\ \vec{c}]$$

Sum Find the volume of a parallelepiped whose edges are $\vec{i}+2\vec{j}+3\vec{k}$, $3\vec{i}+7\vec{j}-4\vec{k}$ and $\vec{i}-5\vec{j}+3\vec{k}$,

Solution The volume of the parallelepiped

$$= (\vec{i}+2\vec{j}+3\vec{k})\cdot(3\vec{i}+7\vec{j}-4\vec{k})\times(\vec{i}-5\vec{j}+3\vec{k})$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix} \qquad [R_1 \to R_1 - R_3]$$

$$= \begin{vmatrix} 0 & 7 & 0 \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix}$$

$$= -7\begin{vmatrix} 3 & -4 \\ 1 & 3 \end{vmatrix} = -7(9+4)$$

$$= -91 \text{ cubic unit.}$$

Since, volume cannot be negative, so, the required volume $= 91$ cubic unit.