M. Sc. (Maths.) sem I

Paper I (CC 01)

By

RINGS

ALOK KUMAR
MAHARAJA COLLEGE, ARA

$(R, +, \cdot) \longrightarrow$ is a ring.

You have already studied it in B.Sc (Hons.)

Just we revise its fundamentals:-

Let R be a non-empty set. Let $(+)$ and $(\cdot)$ be (i.e. addition and multiplication respectively) be two binary operations defined on R.

Then $(R, +, \cdot)$ is called a ring if

(A) $(R, +)$ is an abelian group.

(B) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ [Associative law for multiplication]

(C) $\left.\begin{array}{l} a \cdot (b+c) = a \cdot b + a \cdot c \\ (b+c) \cdot a = b \cdot a + c \cdot a \end{array}\right\} \forall\, a, b, c \in R$

[Distributive law]

EXAMPLES

1. The set of all integers with addition and multiplication as binary operations is a ring.

It is denoted by $(I, +, \cdot)$ or $(Z, +, \cdot)$.

Scanned with CamScanner

* Commutative ring

A ring in which multiplication is commutative,

i.e. $a.b = b.a \ \forall\, a,b \in (R,+,\cdot)$

* Ring with unity

A ring $(R,+,\cdot)$ which contains the multiplicative identity 1 (called unity).

* Null ring or zero ring

The set R which has a single element 0 with two binary operations (+) and (·) defined by

$0+0=0 \quad , \quad 0.0=0$

is a ring, called Null ring/zero ring.

Scanned with CamScanner

Homomorphism of Rings

Let R and R' be two rings.
Let ϕ maps R into R' i.e.

$\phi : R \to R'$.

Then ϕ is called a homomorphism if

(A) $\phi(a+b) = \phi(a)+\phi(b)$
(B) $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ $\forall a,b \in R$

Binary operations of R

Binary operations of R'

i.e. (+) and (·) on the left of (A) and (B) are bi. operations of ER.

and (+) and (·) on the right of A and B are binary operations of R'.

$\boxed{Q}$ If ϕ is a homomorphism of R into R' then

(a) $\phi(0) = 0'$ where $0 \in R$, $0' \in R'$

(b) $\phi(-a) = -\phi(a)$. $\forall a \in R$.

Soln

$\because$ ϕ is a homomorphism

By definition, $\phi(a+b) = \phi(a) + \phi(b)$

$\Rightarrow \phi(a+0) = \phi(a) + \phi(0)$

$\Rightarrow \phi(a) = \phi(a) + \phi(0)$ — (1)

Also, $\phi(0+a) = \phi(0) + \phi(a)$

$\Rightarrow \phi(a) = \phi(0) + \phi(a)$ ——— (2)

From (1) and (2)

$\phi(a) + \phi(0) = \phi(a) = \phi(0) + \phi(a)$

$\Rightarrow$ $\phi(0)$ is the additive identity of R'. i.e. $\phi(0)$ is the zero element of R'.

Let $0'$ = zero element of R'

$\Rightarrow \phi(0) = 0'$. proved (a)

Again $\phi[a + (-a)] = \phi(a) + \phi(-a)$

$\Rightarrow \phi(0) = \phi(a) + \phi(-a)$

$\Rightarrow 0' = \phi(a) + \phi(-a)$

$\Rightarrow \phi(-a) = -\phi(a)$ proved (b).