Blackbody Radiation-Section5

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1 Stefan-Boltzman law of blackbody radiation

Stefan-Boltzman law correlates the power radiated per unit area from a blackbody to its absolute temperature. According to this law, *the energy radiated from a blackbody per unit area per unit time is directly proportional to the fourth power of its absolute temperature.* If E is the energy emitted from a surface area A of a blackbody per second and T is the absolute temperature of the blackbody, then Stefan-Boltzman law predicts:

$$\frac{E}{A} \propto T^4$$

Energy emitted per unit time is Power (P). Power emitted per unit area = $\frac{P}{A}$ = Power density = j. Therefore, Stefan-Boltzman law can be expressed as:

$$j \propto T^4$$

$$j = \sigma T^4 \tag{1}$$

Here, σ is the constant of proportionality, called Stefan-Boltzman's constant. Therefore, the Stefan-Boltzman law may be stated as, *the radiation power density from a blackbody varies with the fourth power of its absolute temperature.*

1.1 Derivation of Stefan-Boltzman law from Planck's law

Planck's law of blackbody radiation states that, the energy density per unit wavelength inside a blackbody cavity at temperature T

$$S_\lambda = \frac{\partial u_\lambda}{\partial \lambda} = \frac{8\pi hc}{\lambda^5(e^{(hc/\lambda k_B T)} - 1)} \tag{2}$$

If we integrate equation (2) over all the possible λ values, then u_λ (energy density integrated over all λ values) may be expressed as:

$$\int_0^\infty S_\lambda d\lambda = u_\lambda = 8\pi hc \int_0^\infty \frac{d\lambda}{\lambda^5(e^{(hc/\lambda k_B T)} - 1)} \tag{3}$$

The total energy $E_\lambda = u_\lambda V$, where $V=L^3$ is the volume of the cavity.

$$\frac{E_\lambda}{V} = u_\lambda = 8\pi hc \int_0^\infty \frac{d\lambda}{\lambda^5(e^{(hc/\lambda k_B T)} - 1)} \tag{4}$$

$$\frac{E_\lambda}{V} = u_\lambda = 8\pi hc \int_0^\infty \frac{1}{(e^{(hc/\lambda k_B T)} - 1)} \frac{1}{\lambda^3} \frac{d\lambda}{\lambda^2} \tag{5}$$

To solve the integration, let's say,

$$x = \frac{hc}{\lambda k_B T} \tag{6}$$

$$\frac{1}{\lambda} = \frac{k_B T}{hc} x \tag{7}$$

Differentiating both side,

$$-\frac{1}{\lambda^2} d\lambda = \frac{k_B T}{hc} dx \tag{8}$$

Using equation (6) and (8) into equation (5):

$$\frac{E_\lambda}{V} = 8\pi ch(\frac{k_B T}{ch})^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \tag{9}$$

Here, we have neglected the '-' sign, since energy can not be negative.
Note: While deriving the Rayleigh-Jeans equation we have seen (see, Blackbody Radiation Section 2) that the number of modes decreases with increasing wavelength. Mathematically, this gives relation between number of modes per unit wavelength per unit volume:

$$\frac{\partial N}{\partial \lambda} = -\frac{8\pi}{\lambda^4}$$

Therefore, S_λ in equation (2) appears with a negative sign. Hence, RHS of equation (9) becomes positive.
The standard integral $\int_0^\infty \frac{x^3 dx}{e^x - 1}$ in equation (9) has solution $\frac{\pi^4}{15}$. Therefore,

$$\frac{E_\lambda}{V} = \frac{8\pi k_B^4 T^4}{(ch)^3} \frac{\pi^4}{15} \tag{10}$$

Though equation (10) has T^4 dependance, it is not the Stefan-Boltzman law which provides the total energy radiated by a black body per unit surface area per unit time. In fact, while deriving the Planck's law, we never consider flow of energy from the cavity. The radiation flow is the same in all directions, and propagates at the speed of light (c). Equation 10 is expressed in terms of the Stefan-Boltzman constant σ by:

$$\frac{E_\lambda}{V} = \frac{4\sigma T^4}{c} \tag{11}$$

Here, $\frac{4\sigma}{c}$ is called the radiation constant. The expression for σ is therefore:

$$\sigma = \frac{2k_B^4 \pi^5}{15c^2 h^3} \tag{12}$$

The numerical value of σ is 5.67×10^{-8} $\mathrm{Wm^{-2}K^{-4}}$.

2 Kirchhoff's law of radiation

Before we state the law let's define emissivity (ϵ) and absorptivity (α).
The emissivity (ϵ) of the surface of a material is its effectiveness in emitting energy as thermal radiation. The measurement of emissitivity is defined by ($\epsilon = \frac{P}{P_B}$) where P is radiated power per unit area from the real body at temperature T, and P_B is radiated power per unit area from a black-body at the same temperature T. The emissivity of the blackbody is 1 and that of a real body is in between 0 and 1.
Absorptivity (α) is the fraction of incident radiation absorbed. A blackbody has absorptivity 1 and absorbs all radiation incident on it.
Kirchhoff's law is defined: *For an arbitrary body emitting and absorbing thermal radiation in thermodynamic equilibrium, the emissivity is equal to the absorptivity.* (Consult any of the following textbooks for proof.)

References

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[1] Figures are collected from online resources.