

Lagrange's linear partial diff. eqns

Here, z is a function of x and y .

$$\text{i.e. } z = z(x, y).$$

$$\therefore p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial x} = z_x, \quad \frac{\partial z}{\partial y} = z_y$$

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Form of Lagrange's equation

$$Pp + Qq = R$$

$$\text{or } P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R$$

Here, P, Q, R are functions of x, y, z .

AUXILIARY EQUATIONS

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (A)}$$

We find two integrals from eq (A)

Let these be $u = a$ and $v = b$.

Then the general integral is

$$f(u, v) = 0 \quad \text{or } \phi(u) = v$$

$$\text{or } u = \phi(v)$$

Examples 1. $p + q = 1$

Compare it with $Pp + Qq = R$

i.e. $P=1, Q=1, R=1$.

∴ Auxiliary equations are

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$$

$$\Rightarrow dx = dy = dz \quad \text{--- (A)}$$

From (A), $dx = dy$. Integrating, we get

$$x = y + u \quad \Rightarrow \quad u = x - y \quad \text{--- (B)}$$

From (B) $dy = dz$

Integrating, we get

$$y = z + v \quad \Rightarrow \quad v = y - z \quad \text{--- (C)}$$

Note that eq. (B) and (C) are two integrals.

Hence, the general solution is

$$u = f(v)$$

$$\Rightarrow \underline{\underline{x - y = f(y - z)}}$$

$$(y+z)p + (z+x)q = x+y.$$

Soln

It is of the form

$$Pp + Qq = R$$

$$\therefore P = y+z, Q = z+x, R = x+y.$$

Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{--- (A)}$$

Students \rightarrow find two integrals from (A).

$$\text{From (A)} \quad \frac{dx-dy}{(y+z)-(z+x)} = \frac{dy-dz}{(z+x)-(x+y)}$$

$$\Rightarrow \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\Rightarrow \frac{dx-dy}{x-y} - \frac{dy-dz}{y-z} = 0$$

Integrating, we get

$$\log(x-y) - \log(y-z) = \log u$$

$$\Rightarrow u = \frac{x-y}{y-z} \quad \text{--- (B)}$$

$$\text{Again, from (A),} \quad \frac{dx-dz}{(y+z)-(x+y)} = \frac{dx+dy+dz}{y+z+z+x+x+y}$$

$$\Rightarrow \frac{dx+dy+dz}{2(x+y+z)} + \frac{dx-dz}{x-z} = 0$$

$$\Rightarrow \frac{dx+dy+dz}{x+y+z} + \frac{2(dx-dz)}{x-z} = 0$$

Integrating, we get

$$\log(x+y+z) + 2 \log(x-z) = \log v$$

$$\Rightarrow \log(x+y+z) + \log\{(x-z)^2\} = \log v$$

$$\Rightarrow \log[(x+y+z)(x-z)^2] = \log v$$

$$\Rightarrow v = (x+y+z)(x-z)^2 \longrightarrow (C)$$

Eq. (B) and (C) are two integrals of the given eqn.

Hence the general soln of the given eqn is

$$u = \phi(v)$$

$$\text{i.e. } \frac{x-y}{y-z} = \phi[(x+y+z)(x-z)^2]$$



Soln

Solve $(a-x)p + (b-y)q = c-z$.
Here, $p = a-x$, $q = b-y$, $R = c-z$
Auxiliary equations are

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{a-x} = \frac{dy}{b-y} = \frac{dz}{c-z} \quad \text{--- (A)}$$

From (A), $\frac{dx}{a-x} = \frac{dy}{b-y}$. Integrating

$$-\log(a-x) = -\log(b-y) - \log u$$

$$\Rightarrow \log(a-x) = \log(b-y) + \log u$$

$$\Rightarrow u = \frac{a-x}{b-y} \quad \text{--- (B)}$$

From (B), $\frac{dx}{a-x} = \frac{dz}{c-z}$ integrating.

$$-\log(a-x) = -\log(c-z) + \log v$$

$$\Rightarrow v = \frac{a-x}{c-z} \quad \text{--- (C)}$$

Q Solve $p \tan x + q \tan y = \tan z$

Soln

Here $p = \tan x$, $q = \tan y$, $R = \tan z$
Auxiliary eqns are

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

So the soln is $f(u,v) = 0$

$$\Rightarrow \cot x dx = \cot y dy = \cot z dz$$

$$\therefore \cot x dx = \cot y dy$$

$$\Rightarrow \log \sin x = \log \sin y + u$$

$$\Rightarrow u = \frac{\sin x}{\sin y}$$

$$\text{Similarly } v = \frac{\sin x}{\sin z}$$

Q Solve

$$zp - zq = z^2 + (x+y)^2$$

Soln

Comparing it with $Pp + Qq = R$, we get

$$P = z, \quad Q = -z, \quad R = z^2 + (x+y)^2$$

\therefore Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2} \quad \text{--- (A)}$$

From (A), $\frac{dx}{z} = \frac{dy}{-z}$

$$\Rightarrow dx + dy = 0. \text{ Integrating, we get}$$

$$\Rightarrow x + y = u \quad \text{--- (B)}$$

From (A), $\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$

$$\Rightarrow dx = \frac{z dz}{z^2 + (x+y)^2} = \frac{z dz}{z^2 + u^2} \quad \left[\begin{array}{l} \text{using} \\ \text{(B)} \end{array} \right]$$

Integrating, we get

$$\Rightarrow dx = \log(z^2 + u^2) - v$$

$$\Rightarrow v = \log\{z^2 + (x+y)^2\} - 2x \quad \text{--- (C)}$$

Hence, the general integral is

$$v = \psi(u) \Rightarrow \log\{z^2 + (x+y)^2\} - 2x = \psi(x+y)$$