Quantum Mechanics-Section6

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1 Particle nature of waves

1.1 Compton Effect

Compton effect is another significant experimental evidence that supports the quantization of light. Moreover, this study performed by A. H. Compton shows that the wave property of light may be explained through its particle-like behaviour. Compton effect therefore establishes a direct correlation between light's wave and particle nature.

In 1923 Compton was experimenting the scattering of x-ray of known wavelength on graphite while he measured change in wavelength $\Delta\lambda$ in the scattered x-ray. According to his observation, $\Delta\lambda$ depends on the angle of scattering ϕ.

This observation is however peculiar from the viewpoint of classical mechanics. Because, according to classical theory, when electromagnetic wave of fixed frequency ν falls on the scatterer or target material, the electrons in the target oscillate with same frequency ν. Therefore, these oscillating charged particles (electrons) radiate electromagnetic waves of same frequency ν and therefore the scattered waves will have same wavelength λ as that of the incident wave.

To explain the observed phenomena, Compton used simple theory of elastic collision, assuming that light is made up of energy corpuscles called 'photon'. These photons collide with the electrons inside scattering material. As a result of collision, the photons scatter off and the electrons recoil back at different angles. In elastic collision, the net momentum and total kinetic energy of the colliding particles remains conserved. Compton used this basic logic of classical theory of collision to calculate the change of total energy of the photon.

The scenario before and after collision is shown in figure (1). The electron is assumed to be at rest before collision, while the photon approaches towards the electron and eventually collides with it. After collision, the scattered photon moves making an angle ϕ with the x-axis, while the electron recoils at angle θ.

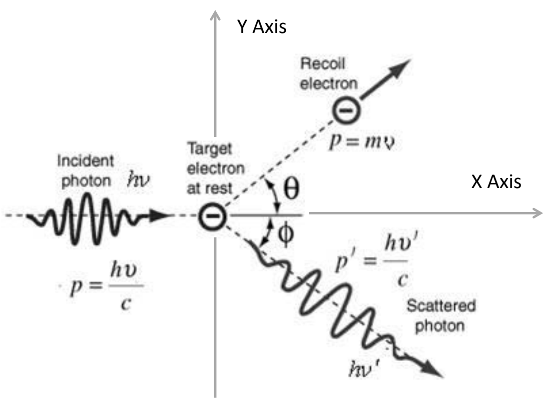

Figure 1: Compton Scattering: The scattered x-ray has higher wavelength λ' compared to the incident wavelength λ.

We have to calculate the energies and linear momenta of the colliding particles before and after collision. Let's assume the photon has wavelengths λ and λ' before and after collision, respectively. The photon has total energy $h\nu = \frac{hc}{\lambda}$ and $h\nu' = \frac{hc}{\lambda'}$ before and after collision, respectively. The electron is at rest before collision. Therefore, its kinetic energy is zero. After collision, it moves with a velocity v. The corresponding momentum $p = mv$. If we consider electron rest mass and recoil mass to be m_0 and m, respectively then it has rest mass energy and relativistic recoil energy $E_0 = m_0 c^2$ and $E = mc^2 = (p^2c^2 + m_0^2 c^4)^{\frac{1}{2}}$, respectively. Considering conservation of energy:

$$\frac{hc}{\lambda} + m_0 c^2 = \frac{hc}{\lambda'} + mc^2 \tag{1}$$

$$\frac{hc}{\lambda} + m_0 c^2 = \frac{hc}{\lambda'} + \sqrt{p^2c^2 + m_0^2 c^4} \tag{2}$$

$$(\frac{hc}{\lambda} - \frac{hc}{\lambda'}) - m_0 c^2 = \sqrt{p^2c^2 + m_0^2 c^4} \tag{3}$$

Squaring both sides:

$$\left((\frac{hc}{\lambda} - \frac{hc}{\lambda'}) - m_0 c^2\right)^2 = p^2c^2 + m_0^2 c^4 \tag{4}$$

$$\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)^2 + m_0^2 c^4 - 2m_0 c^2 \left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) = p^2c^2 + m_0^2 c^4 \tag{5}$$

Subtracting $m_0^2 c^4$ from both sides:

$$\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)^2 - 2m_0 c^2 \left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) = p^2c^2 \tag{6}$$

$$p^2c^2 = \left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - 2\frac{hc}{\lambda}\frac{hc}{\lambda'} - 2m_0 c^2 \left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) \tag{7}$$

Now, we have to consider the momentum conservation. The conservation of linear momentum will be considered along their x and y-components. For the case of incident photon, the whole momentum $\frac{h}{\lambda}$ is along the x-direction. The electron has zero initial momentum. On the other hand, scattered photon has momentum $\frac{h}{\lambda'}$. Its component along x-direction is $\frac{h}{\lambda'} cos\phi$. Similarly, the recoil electron has momentum p, whose x-component, according to figure (1), is $p\, cos\theta$. Therefore, momentum conservation along x-axis gives:

$$\frac{h}{\lambda} = p\, cos\theta + \frac{h}{\lambda'} cos\phi \tag{8}$$

$$p\, cos\theta = \frac{h}{\lambda} - \frac{h}{\lambda'} cos\phi \tag{9}$$

Now let us consider the conservation of momentum along the y-axis. The incident photon has no momentum component along y-axis. The electron at rest has zero momentum. After collision, the recoil electron has y-component of its momentum $p\, sin\theta$. Similarly, the photon's y-component of its momentum is $-\frac{h}{\lambda'} sin\phi$ ('-' sign appears due to motion of the photon along negative y-axis). From the conservation of linear momentum we write:

$$0 = -\frac{h}{\lambda'} sin\phi + p\, sin\theta \tag{10}$$

$$p\, sin\theta = \frac{h}{\lambda'} sin\phi \tag{11}$$

$$p^2 = p^2 sin^2\theta + p^2 cos^2\theta \tag{12}$$

Using equation (9) and (11):

$$p^2 = \left(\frac{h}{\lambda'} sin\phi\right)^2 + \left(\frac{h}{\lambda} - \frac{h}{\lambda'} cos\phi\right)^2 \tag{13}$$

$$p^2c^2 = \left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}cos\phi \tag{14}$$

Now, we compare equation (7) and (14), which gives:

$$\left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - 2\frac{hc}{\lambda}\frac{hc}{\lambda'} - 2m_0c^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) = \left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}cos\phi \tag{15}$$

$$2\frac{hc}{\lambda}\frac{hc}{\lambda'}(1 - cos\phi) = 2m_0c^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) \tag{16}$$

$$(1 - cos\phi) = \frac{m_0c}{h}(\lambda' - \lambda) \tag{17}$$

Change in wavelength is defined as: $\lambda' - \lambda = \Delta\lambda$

$$\Delta\lambda = \frac{h}{m_0c}(1 - cos\phi) \tag{18}$$

The quantity $\lambda_c = \frac{h}{m_0c} = 0.0243$ Å is called the *Compton wavelength*. $\Delta\lambda$ is the Compton shift, it depends on the scattering angle ϕ, not on the initial wavelength. From equation (18), the wavelength shift $\Delta\lambda$ is zero for $\phi = 0$, *i.e.* for grazing collision. The maximum value for the Compton shift is $\frac{2h}{m_0c}$, which is obtained for $\phi = \pi$, *i.e.* for head-on collision.

In derivation of equation (18), we have assumed that the electron, which takes part in collision, is free before the impact. However, it does not happen for all practical cases. In most of the cases, the electron remains bound to some atom. In those cases, the approximation of free electron is valid, only when the binding energy of the electron is ignorable compared to the kinetic energy of the electron after collision. This happens when the incident photon has high frequency, hence, the term $h\nu$ is no longer small enough to be negligible. On the other hand, photons with low frequency (and therefore of low energy) can not tear off the binding of the electron to its atomic nucleus. In these cases, the collision of the photon may be assumed not with the electron, but with the whole atom. Therefore, in equation (18), the electron mass m_0 should be replaced by the atomic mass M. Here $M >> m_0$, therefore, the change in wavelength $\Delta\lambda$ is negligible. In fact, there occurs no change in wavelength in such cases. The process that scatters photons without changing their wavelength is called *Rayleigh Scattering*.

At lower wavelength limit $\lambda \to 0$ (theoretically), the scattering phenomena is Compton scattering and may be explained by quantum mechanics. This happens mostly for the case of x-ray and gamma rays. At higher wavelength limit $\lambda \to \infty$, the scattering phenomena is Rayleigh scattering and may be explained by classical mechanics. This happens for the case of infrared, radio-wave *etc.*

References

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[1] Figures are collected from online resources.