M. Sc. (Mathematics) Sem. I 4th Paper

DISCRETE MATHS. $\boxed{CC-04}$

Partially ordered set (or POSET) (or POS)

Let S be any non-empty set.
Let R be a relation defined on S.
If R satisfies the following properties:

(i) R is reflexive i.e. $xRx \ \forall x \in S$

(ii) R is anti symmetric
i.e. $xRy, yRx \Rightarrow x = y \ \forall x, y \in S$

(iii) R is transitive $\forall x, y, z \in S$.
i.e. $xRy, yRz \Rightarrow xRz$

then R is called a partial order relation (POR) and the combination of the sets S ~~set~~ and relation R is called a Partially ordered set, denoted by (S, R).

A Partial order relation is symbolized by $\leq$.

Examples I. Set N = Set of natural numbers $= \{1, 2, 3, ---\}$

and relation $R = \leq$ (less than or equal to)

$\therefore (N, \leq)$ is a partially ordered set

because (i) $x \leq x \ \forall x \in N$

(ii) $x \leq y, y \leq x \Rightarrow x = y \ \forall x, y \in N$

(iii) $x \leq y, y \leq z \Rightarrow x \leq z \ \forall x, y, z \in N$

2. N = set of natural numbers

R = x divides y where $x, y \in N$

(N, R) is a partially ordered set because

(i) 2 divides 2, i.e. x divides x $\forall x \in N$

(ii) x divides y, y divides $x \Rightarrow x = y$

i.e. $x \mid y,\ y \mid x \Rightarrow x = y$

(iii) If x divides y, y divides $z \Rightarrow x$ divides z

e.g. 2 divides 12, 12 divides 24 $\Rightarrow$ 2 divides 24

i.e. $x \mid y,\ y \mid z \Rightarrow x \mid z$.

3. Let P = collection of all subsets of some universal set U and

$R = \subseteq$ i.e. $A \leq B$ means $A \subseteq B$.

Then $(P, \subseteq)$ is a partially ordered set.

4. Let P = set of all real functions defined defined on a non-empty set X

and let $f \leq g$ mean that $f(x) \leq g(x)\ \forall x \in X$

Then $(P, \leq)$ is a partially ordered set.

Comparable elements

Two elements in a POSET are called comparable ; if one of them is less than or equal to the other that is, either $x \leq y$ or $y \leq x$.

A partial order relation with comparable property is called a Total (or linear) order relation and such a POSET is called a totally ordered set, or a linearly ordered set.

Maximal element

Let P be a partially ordered set. An element x in P is said to be maximal if $y \geq x \Rightarrow y = x$, that is, no element other than x itself is greater than or equal to x.

A maximal element in P is thus an element of P which is not less than or equal to any other element of P.

* Let A be a non-empty set of a partially ordered set P.

An element x in P is called a LOWER BOUND of A if $x \leq a \; \forall \; a \in A$

and a lower bound of A is called a greatest lower bound (glb) of A if it is greater than or equal to every lower bound of A.

Again, an element y in P is said to be an upper bound of A if

$a \leq y \; \forall \; a \in A$

and a least upper bound (lub) of A is an upper bound of A which is less than or equal to every upper bound of A.

* ~~A~~ The set A may have many lower bounds and many upper bounds.

* A greatest lower bound or least upper bound is unique if it exists.