Quantum Mechanics-Section11

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1 Properties of matter waves

The velocity of propagation of an wave is $v_w = \lambda\nu$. Using de-Broglie equation $E = h\nu$ and $p = \frac{h}{\lambda}$,

$$v_w = \lambda\nu = \frac{h}{p}\frac{E}{h} = \frac{E}{p}$$

If the particle moves with non-relativistic velocity v_p, then

$$v_w = \frac{E}{p} = \frac{mv_p^2/2}{mv_p} = \frac{v_p}{2}$$

It may appear that the wave (with velocity v_w) can not keep up with the particle (v_p). However this is not the case. To understand this puzzle one need to realize the configuration of matter-wave corresponding to a particle. Let us assume that there is a particle at rest at point x in a one-dimensional space. Suppose, we replace this particle by a wave, using quantum mechanical view-point, which looks like the wavetrain as shown in figure (1).

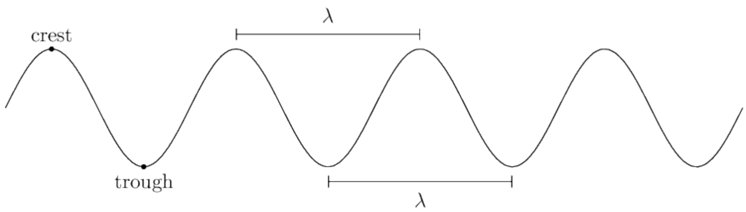

Figure 1: A theoretical wavetrain.

However, this will be a foolish idea. There are several reasons for it. Firstly, a theoretical wavetrain is infinitely extended. In space (one dimensional), it extends from $-\infty$ to $+\infty$, which may intend us to think that the corresponding particle may be anywhere in space. However, the particle, we are discussing about, is at rest at a point x. How can an infinitely extended wave correspond to a localized entity then? Simply, it can not. Therefore, the corresponding matter wave of a particle has to be finite in space. Secondly, the matter-wave should have maximum amplitude at the exact location of the particle. The amplitude decreases significantly as we move towards either side from the exact location of the particle. In fact, in quantum mechanics, exact location of a quantum particle is vague, the particle location may be determined within a region only. Therefore, we can identify a region in space where the probability of finding the particle is maximum. Within that region, the corresponding matter-wave amplitude is large. So, the matter-wave corresponding to a particle shall look like the figure shown in (2)

A wavepacket is basically a group of waves. If the particle moves along the x axis the corresponding wave-packet also moves with the particle and it is represented mathematically by $\psi(x,t)$.

First, let us discuss the simplest type of wave motion, a sinusoidal wave moving towards positive x-axis. It may be written in the following form:

$$f(x,t) = sin\,(kx - \omega t)$$

This is a pure sinusoidal function with varying amplitude. It has periodical nodes corresponding to

$$kx - \omega t = n\pi$$

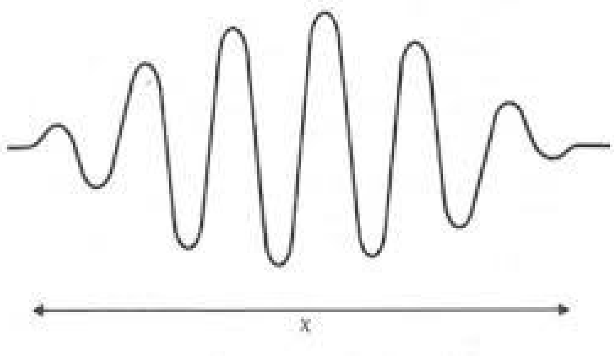

Figure 2: A wavepacket - represents a matter wave.

Therefore,

$$x = \frac{1}{k}(n\pi + \omega t)$$

The velocity of the sinusoidal wave v_w may be expressed as:

$$v_w = \frac{dx}{dt} = \frac{\omega}{k}$$

We shall now discuss the formation of wave group, where the amplitude is modulated. A wave group is similar as shown in figure (2) propagating along the x-direction. Let's from a wavegroup mathematically, by adding just two sinusoidal waves to keep it simple. If a wavegroup $\psi(x,t)$ is created by combination of two sinusoidal waves $f_1(x,t)$ and $f_2(x,t)$ propagating along the x-direction.

$$\psi(x,t) = f_1(x,t) + f_2(x,t)$$

where,

$$f_1(x,t) = sin\,(kx - \omega t)$$

and

$$f_2(x,t) = sin\,((k+dk)x - (\omega + d\omega)t)$$

Hence, we get the final form of the wave group as:

$$\psi(x,t) = 2cos\,\left(\frac{dk}{2}x - \frac{d\omega}{2}t\right) sin\,(kx - \omega t)$$

The second part of the *rhs* represents its wave motion along the x-axis. The first part implies the modulated amplitude of wave group. Both the wave group and the waves it contains propagates along the x direction. The velocity v_w of individual wave may be calculated from the second term and the velocity v_g of the wave group may be obtained from the first term.

$$v_w = \frac{\omega}{k}$$

$$v_g = \frac{d\omega/2}{dk/2} = \frac{d\omega}{dk}$$

. We are now in a position to calculate group velocity of matter waves associated to a moving particle.

$$v_g = \frac{d\omega}{dk}$$

since, $E = h\nu$, $\nu = fracEh$, Therefore $\omega = 2\pi\nu = \frac{2\pi E}{h}$ and $\lambda = \frac{h}{p}$, $k = \frac{2\pi}{\lambda}$, Therefore, $k = \frac{2\pi p}{h}$.

$$d\omega = \frac{2\pi dE}{h}$$

$$dk = \frac{2\pi dp}{h}$$

$$\left(\frac{d\omega}{dk}\right) = \frac{\frac{2\pi dE}{h}}{\frac{2\pi dp}{h}}$$

$$\left(\frac{d\omega}{dk}\right) = \frac{dE}{dp}$$

Setting $E = \frac{1}{2}mv_p^2$, and $p = mv_p$,

$$\left(\frac{d\omega}{dk}\right) = \frac{dE}{dp} = \frac{mv_p dv_p}{m dv_p} = v_p$$

This proves that the group velocity is equal to the particle velocity. Therefore the matter wave group moves with the particle.

References

[1] Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, Authors: R. Eisberg, R. Resnick, Publisher: Wiley

[2] Quantum Physics, Author: Stephen Gasiorowicz, Publisher: John Wiley and Sons

[3] Quantum Mechanics: Theory and Applications, Authors: A. Ghatak, S. Lokanathan, Publisher: Trinity Press

[4] Modern Quantum Mechanics, Author: J.J. Sakurai, Publisher: Benjamin/Cummings Publisher.

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[1] Figures are collected from online resources.