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B.Sc. Part I (Hons) 2nd Paper Diff. Calculus

Successive Differentiation continued

1. If $y = (\sin^{-1}x)^2$, prove that

$(1-x^2)y_2 - xy_1 - 2 = 0$ and

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0.$

Soln $\because y = (\sin^{-1}x)^2$ ——— (1)

Differentiating with respect to x, we get

$y_1 = 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$ squaring both sides,

$\Rightarrow (1-x^2)y_1^2 = 4(\sin^{-1}x)^2$

$\Rightarrow (1-x^2)y_1^2 - 4y = 0$ [using (1)]

Differentiating w.r.to x, we get

$\Rightarrow (1-x^2)\cdot 2y_1y_2 - 2xy_1^2 - 4y_1 = 0$

Dividing by $2y_1$, we get

$\Rightarrow (1-x^2)y_2 - xy_1 - 2 = 0$ [1st part proved]

Differentiating n times the above eqn, with respect to x, by Leibnitz's theorem, we get

$[y_2(1-x^2)]_n - [y_1 x]_n - [2]_n = 0$

$$\Rightarrow \left[y_2\right]_n (1-x^2) + {}^nC_1 \left[y_2\right]_{n-1} \cdot \left[1-x^2\right]_1 + {}^nC_2 \left[y_2\right]_{n-2} \left[1-x^2\right]_2$$

$$+ 0 - \left[y_1\right]_n x - {}^nC_1 \left[y_1\right]_{n-1} \left[x\right]_1 - 0 - 0 = 0$$

$$\Rightarrow y_{n+2} (1-x^2) + n y_{n+1} \cdot (-2x) + \frac{n(n-1)}{\cancel{2}} \times y_n \times (-\cancel{2})$$

$$- y_{n+1} x - n y_n \cdot 1 = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + y_n \left[-n(n-1) - n\right] = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$$

[2nd part proved]

2. If $y = \sin(m \sin^{-1} x)$, prove that

$(1-x^2) y_2 - x y_1 + m^2 y = 0$ and

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 - m^2) y_n = 0.$

Soln

$$y = \sin(m \sin^{-1} x) \quad \text{—— (1)}$$

Differentiating it with respect to x, we get-

$$\Rightarrow y_1 = \cos(m \sin^{-1} x) \times \frac{d}{dx}(m \sin^{-1} x)$$

$$\Rightarrow y_1 = \cos(m \sin^{-1} x) \times \frac{m}{\sqrt{1-x^2}}$$

$\Rightarrow \sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$

Squaring both sides,

$\Rightarrow (1-x^2)\, y_1^2 = m^2 \cos^2(m\sin^{-1}x)$

$= m^2\left[1-\sin^2(m\sin^{-1}x)\right] = m^2(1-y^2)$

[using (1)]

$\Rightarrow (1-x^2)\, y_1^2 = m^2 - m^2 y^2$

Differentiating it with respect to x, we get

$\Rightarrow (1-x^2)\, 2y_1 y_2 - 2x\, y_1^2 = 0 - m^2\, 2y y_1$

Dividing by $2y_1$, we have

$(1-x^2)\, y_2 - x y_1 + m^2 y = 0$ [1st part proved]

Differentiating n times, the above eqn with respect to x, we get

$\left[y_2(1-x^2)\right]_n - \left[y_1 x\right]_n + m^2 \left[y\right]_n = 0$

$\Rightarrow y_{n+2}(1-x^2) + {}^nc_1 \left[y_2\right]_{n-1}\left[1-x^2\right]_1 + {}^nc_2\left[y_2\right]_{n-2}\left[1-x^2\right]_2 + 0$

$- \left[y_1\right]_n x - {}^nc_1\left[y_1\right]_{n-1}\left[x\right]_1 - 0 + m^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} + n\, y_{n+1}\cdot(-2x) + \frac{n(n-1)}{\cancel{2}} \times y_n \times (-\cancel{2})$

$- y_{n+1} x - n\, y_n \cdot 1 + m^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} - (n^2 - m^2)\, y_n = 0$

[2nd ~~part~~ proved]

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3. If $y = e^{a\sin^{-1}x}$, prove that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$

Soln $\because y = e^{a\sin^{-1}x}$ ——— (1)

Differentiating it with respect to x, we get

$y_1 = e^{a\sin^{-1}x} \times \frac{d}{dx}(a\sin^{-1}x) = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}}$

$\Rightarrow y_1 = \frac{ay}{\sqrt{1-x^2}}$ [using (1)]

$\Rightarrow \sqrt{1-x^2}\, y_1 = ay \rightarrow \left(\sqrt{1-x^2}\, y_1\right)^2 = a^2y^2$

$\Rightarrow (1-x^2)y_1^2 = a^2y^2$

Differentiating ~~[illegible]~~ with respect to x, by

~~[illegible]~~ we get

$\cancel{\Rightarrow \left[y_1^2(1-x^2)\right]_n = a^2\left[y^2\right]_n}$

$\Rightarrow (1-x^2)2y_1y_2 - 2xy_1^2 = a^2 \cdot 2yy_1$

Dividing by $2y_1$, we get

$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$. Diff. n times by Leibnitz

$\Rightarrow y_{n+2}(1-x^2) + n\, y_{n+1} \cdot (-2x) + \frac{n(n-1)}{\cancel{2}} y_n \cdot (-\cancel{2})$

$- y_{n+1}x - n y_n - a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$