Quantum Mechanics-Section 12

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1 Formation of localized wave packets

a localized wave packet can be achieved, by superposing waves of different frequencies in a special way. These waves interfere completely with each other within a given spatial region. Mathematical tool to achieve this fact is the application of Fourier Integrals. As an example, consider the function defined by:

$$f(x) = \int_{-\infty}^{\infty} dk\, g(k) e^{ikx}$$

The real part of $f(x)$ is given by $\int_{-\infty}^{\infty} dk\, g(k) cos\, kx$, and this is a linear superposition of waves of wavelength $\lambda = \frac{2\pi}{k}$, since for a given k each wave reproduces itself when x changes to $x + \frac{2\pi}{k}$. To illustrate such a wave packet, let us choose

$$g(k) = e^{\alpha(k-k_0)^2}$$

The integral can be done with $k' = k - k_0$ we have

$$f(x) = e^{ik_0 x} \int_{-\infty}^{\infty} dk'\, e^{ik'x} e^{-\alpha k'^2}$$

$$f(x) = e^{ik_0 x} \int_{-\infty}^{\infty} dk'\, e^{-\alpha[k' - (ix/2\alpha)]^2} e^{-(x^2/4\alpha)}$$

It is justified to let $k' - (ix/2\alpha) = q$ and keep the integral along the real axis. Making use of

$$\int_{-\infty}^{\infty} dk\, e^{\alpha\, k^2} = \sqrt{\frac{\pi}{\alpha}}$$

we obtain

$$f(x) = \sqrt{\frac{\pi}{\alpha}} e^{ik_0 x} e^{-(x^2/4\alpha)}$$

The factor $e^{ik_0 x}$ is known as the phase factor, since $|e^{ik_0 x}|^2 = 1$. Thus the absolute square of $f(x)$ is

$$|f(x)|^2 = \frac{\pi}{\alpha} e^{-x^2/2\alpha}$$

This function peaks at $x = 0$ and depending on the magnitude of α, it represents a broad (α large) or narrow (α small) wave packet. $|f(x)|^2$ is representation of a particle. The width of the packet may be taken to be $2\sqrt{2\alpha}$, since the function falls of exponentially to 1/e of its peak value. The width of $|f(x)|^2$ and $|g(k)|^2$ are correlated. The square of $g(k)$ is a function peaked about k_0 and width $\frac{2}{\sqrt{2\alpha}}$. Therefore, a function strongly localized in x is broad in k and vice versa. The product of the two 'widths' is:

$$\Delta\, k \Delta\, x \approx \frac{2}{\sqrt{2\alpha}} .2\sqrt{2\alpha} = 4$$

The exact value of the numerical constant is not important; what matters is that it is independent of α and of the order of unity. Therefore, clearly, it is impossible to make both $\Delta\, x$ and $\Delta\, k$ small.

References

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[1] Figures are collected from online resources.