M. Sc. Sem I Paper I (MAT CC 01)

Unit 2 Abstract Algebra (continued.)

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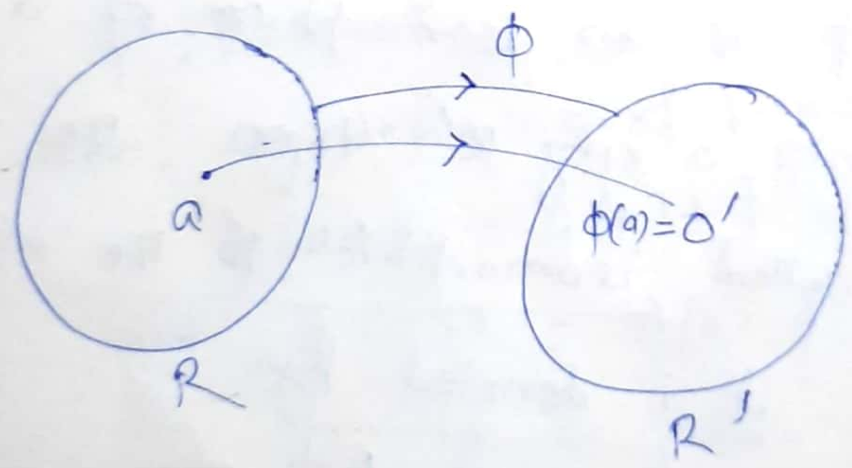
Kernel of Ring homomorphism

Let R and R' be two rings.

Let $\phi : R \to R'$ be a homomorphism.

Then the kernel of homomorphism ϕ, (simply called kernel of ϕ) is the set of all those elements $a \in R$ such that $\phi(a) = 0'$ (zero element of R')

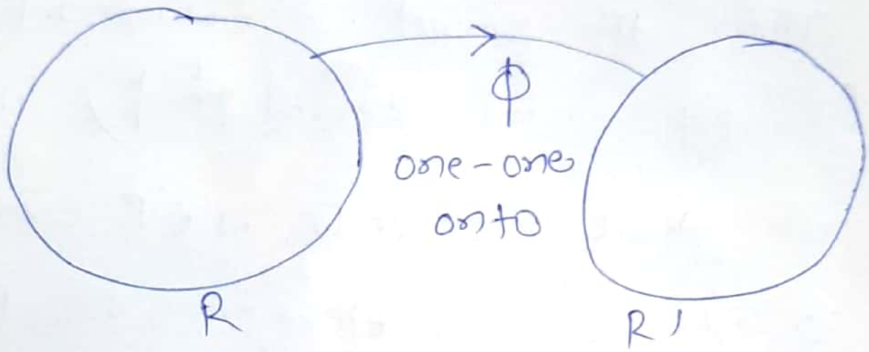


Isomorphism ~~Homomorphism~~ of rings

Let R and R' be two rings.

Let $\phi : R \to R'$ be a homomorphism.

If ϕ is a $\boxed{\text{one-one}}$, $\boxed{\text{onto}}$ mapping then ϕ is called isomorphism.



If ϕ is an isomorphism of a ring R onto a ring R' then the ring R is called isomorphic to the ring R' and it is denoted by

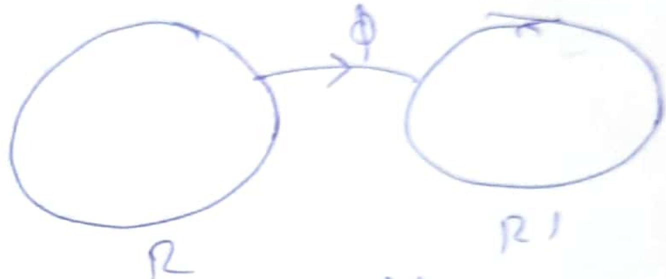
$$\boxed{R \cong R'.}$$

__Theorem__ If ϕ is a homomorphism of ring R into R' and if $K(\phi)$ denotes the kernel of ϕ then prove that

(i) $K(\phi)$ is an additive subgroup of R

(ii) If $a \in K(\phi)$ and $b \in R$ then both ab and ba are in $K(\phi)$.

__Proof__



Given that

$\phi : R \to R'$ is a homomorphism.

then, by definition of homomorphism, we've

$$\phi(a+b) = \phi(a) + \phi(b), \forall a, b \in R \quad -(1)$$

and $$\phi(a \cdot b) = \phi(a) \cdot \phi(b) \ \forall \ a, b \in R.$$

$\Rightarrow$ From (1) it is clear that ϕ is a homomorphism of the additive group $(R, +)$ into another additive group $(R', +)$.

we know that kernel of homomorphism of groups is a subgroup.

$\Rightarrow$ $K(\phi)$ is an additive subgroup of R.

[1st part proved]

Let $a \in K(\phi)$.

Then, by definition of kernel,

$\phi(a) = 0'$, where

$0'$ is the zero-element of R'.

Let $b \in R$.

Now, $\phi(a.b) = \phi(a).\phi(b)$

$= 0'.\phi(b) = 0'$

$\Rightarrow ab \in K(\phi)$

Again, $\phi(b.a) = \phi(b).\phi(a)$

$= \phi(b).0'$

$= 0'$

$\Rightarrow ba \in K(\phi)$

Hence, ab and ba, both are in kernel of ϕ.

Theorem If ϕ be an isomorphism of ring R onto a ring R' then prove that

$\phi(0) = 0'$ and $\phi(-a) = -\phi(a)$

where $0 \in R$, $0' \in R'$, $\forall a \in R$.

Soln $\because \phi$ is a isomorphism

$\Rightarrow \phi$ is necessarily a homomorphism.

In the previous theorem, the above are already proved.