$\boxed{Q.}$ Test for convergence, the series for which

$$U_n = \sqrt{n^4+1} - \sqrt{n^4-1}.$$

Soln

Here, $U_n = \sqrt{n^4\left(1+\frac{1}{n^4}\right)} - \sqrt{n^4\left(1-\frac{1}{n^4}\right)}$

$$= n^2\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}} - n^2\left(1-\frac{1}{n^4}\right)^{-\frac{1}{2}}$$

$$= n^2\left[\cancel{1}+\frac{1}{2n^4} - \cancel{\frac{1}{8n^8}} + \frac{1}{16n^{12}} - \right] - n^2\left[\cancel{1} - \frac{1}{2n^4} - \cancel{\frac{1}{8n^8}} + \cdots\right]$$

$$= n^2\left[\frac{1}{n^4} + \frac{1}{8n^{12}} + \cdots\right]$$

$$\Rightarrow U_n = \frac{1}{n^2} + \frac{1}{8n^{10}} + \cdots$$

Let $V_n = \frac{1}{n^2}$.

$$\therefore \frac{U_n}{V_n} = 1 + \frac{1}{8n^8} + \cdots$$

$$\Rightarrow \lim_{n\to\infty} \frac{U_n}{V_n} = 1$$ which is finite and non-zero.

But $\sum V_n = \sum \frac{1}{n^2}$ is convergent as $p = 2$ here.

$\therefore \sum U_n$ i.e. the given series is also convergent.

[Q] Test, for convergence, the series
$$\sum U_n = \sum \left[\sqrt{n^3+1} - \sqrt{n^3}\right].$$

Soln. Here, $U_n = \sqrt{n^3\left(1+\frac{1}{n^3}\right)} - n^{3/2}$

$$= n^{\frac{3}{2}}\left[\left(1+\frac{1}{n^3}\right)^{\frac{1}{2}} - 1\right]$$

$$\Rightarrow U_n = n^{\frac{3}{2}}\left[\left(1+\frac{1}{2n^3} - \frac{1}{8n^6} + \frac{1}{16n^9} - \cdots\right) - 1\right]$$

$$= n^{\frac{3}{2}}\left[\frac{1}{2n^3} - \frac{1}{8n^6} + \cdots\right]$$

$$\Rightarrow U_n = \frac{1}{2n^{3/2}} - \frac{1}{8n^{9/2}} + \cdots \quad \text{——(1)}$$

consider another series $\sum V_n$ with $V_n = \frac{1}{n^{3/2}}$.

$$\therefore \frac{U_n}{V_n} = \frac{1}{2} - \frac{1}{8n^3} + \cdots$$

$$\Rightarrow \lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{2} \quad \text{which is finite.}$$

So, both series behave alike.

$\therefore \sum V_n = \sum \frac{1}{n^{3/2}}$ is convergent as $p = \frac{3}{2}$.

Hence, the given series $\sum U_n$ is also convergent.

Q. Test for convergence

$$\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots$$

Soln:

The given series

$$=\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots+\frac{2n-1}{n(n+1)(n+2)}+\cdots$$

Here, the general term $U_n=\frac{2n-1}{n(n+1)(n+2)}$.

Let $V_n=\frac{n}{n^3}=\frac{1}{n^2}$ be the general term of another auxiliary series.

$$\therefore \frac{U_n}{V_n}=\frac{n^2(2n-1)}{n(n+1)(n+2)}=\frac{n(2n-1)}{(n+1)(n+2)}$$

$$=\frac{\frac{n(2n-1)}{n^2}}{\frac{(n+1)(n+2)}{n^2}}$$

$$=\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$\Rightarrow \lim_{n\to\infty}\frac{U_n}{V_n}=\frac{2}{1\times1}=2$, which is non-zero and finite.

So, both series converge or diverge together.

But $\sum V_n=\sum\frac{1}{n^2}$ is convergent as $p=2$.

$\Rightarrow \sum U_n$ i.e. the given series is also convergent.

Q. Test for convergence, the series

$$\sum U_n = \sum \sin\frac{\alpha}{n}.$$

__Soln.__ $\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \cdots$

$U_n =$

$$\Rightarrow \sin\frac{\alpha}{n} = \frac{\alpha}{n} - \frac{\alpha^3}{\lfloor 3\, n^3} + \frac{\alpha^5}{\lfloor 5\, n^5} - \cdots$$

Let $V_n = \frac{1}{n}$ be the general term of another auxiliary series.

$$\Rightarrow \frac{U_n}{V_n} = \alpha - \frac{\alpha^3}{\lfloor 3\, n^2} + \frac{\alpha^5}{\lfloor 5\, n^4} - \cdots$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \alpha - 0 + 0 - \cdots = \alpha$$

which is non-zero and finite.

So, both series $\sum U_n$ and $\sum V_n$ behave alike.

Now $\sum V_n = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent.

Hence, $\sum U_n = \sum \sin\frac{\alpha}{n}$ is also divergent.