Quantum Mechanics-Section17

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1 Expectation Values

When the probability density $P(x,t)$ is known, the expectation values of any function $f(x)$ may be calculated as follows:

$$< f(x) >= \int dx\, f(x)\, P(x,t) = \int dx\, \psi^*(x,t)\, f(x)\, \psi(x,t)$$

The integral is only defined under convergence. Also, we shall assume that the wavefunction $\psi(x)$ and all of their derivatives vanish sufficiently rapidly at infinity.

The momentum may be expressed in terms of x, this helps us to calculate the expectation value of momentum. As we know classically

$$p = mv = m\frac{dx}{dt}$$

We may write

$$< p >= m\frac{d}{dt} < x >= m\frac{d}{dt}\int dx\, \psi^*(x,t)x\psi(x,t)$$

This gives

$$< p >= m\int_{-\infty}^{\infty} dx\, \left(\frac{\partial\psi^*}{\partial t}x\psi + \psi^* x\frac{\partial\psi}{\partial t}\right)$$

Note that there is no $\frac{dx}{dt}$ term under the integral sign. The only quantity that varies with time is $\psi(x,t)$, and it is this variation that gives rise to a change in $< x >$ with time. As we know

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}$$

using this equation we obtain:

$$< p >= \frac{\hbar}{2i}\int_{-\infty}^{\infty} dx\, \left(\frac{\partial^2\psi^*}{\partial x^2}x\psi - \psi^* x\frac{\partial^2\psi}{\partial x^2}\right)$$

Now,

$$\frac{\partial^2\psi^*}{\partial x^2}x\psi = \frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}x\psi\right) - \frac{\partial\psi^*}{\partial x}\psi - \frac{\partial\psi^*}{\partial x}x\frac{\partial\psi}{\partial x}$$

$$\frac{\partial^2\psi^*}{\partial x^2}x\psi = \frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}x\psi\right) - \frac{\partial}{\partial x}(\psi^*\psi) + \psi^*\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}\left(\psi^* x\frac{\partial\psi}{\partial x}\right) + \psi^*\frac{\partial\psi}{\partial x} + \psi^* x\frac{\partial^2\psi}{\partial x^2}$$

Hence the integrand has the form

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}x\psi - \psi^* x\frac{\partial psi}{\partial x} - \psi^*\psi\right) + 2\psi^*\frac{\partial\psi}{\partial x}$$

The integral of the derivatives vanishes for square integrable functions, therefore the first term vanishes under integration. Hence,

$$< p >= \int dx\, \psi^*(x,t)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x,t)$$

This suggests that the momentum is represented by operator

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

With the same logic

$$< p^2 >= \int dx\, \psi^*(x,t) \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \right) \psi(x,t)$$

References

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[1] Figures are collected from online resources.