D' Alembert's {Ratio test} B.Sc. Part II (H) Paper IIIrd. Convergency contd.

$$\sum \frac{U_n}{U_{n+1}} > 1 \longrightarrow \sum U_n \text{ is convergent.}$$

$$\leq 1 \Rightarrow \sum U_n \text{ is divergent}$$

<1> Test the convergence of

$$1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \cdots$$

Soln The given series

$$= 1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \cdots$$

Here, $U_n = \dfrac{n^p}{n!} \Rightarrow U_{n+1} = \dfrac{(n+1)^p}{(n+1)!}$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{n^p}{n!} \times \frac{(n+1)!}{(n+1)^p}$$

$$= \frac{(n+1)\, n! \; n^p}{n! \,.\, (n+1)^p}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{\cancel{8}\cdot(n+1)}{\left(1+\frac{1}{n}\right)^p}$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt}\left[\frac{n+1}{\left(1+\frac{1}{n}\right)^p}\right] = \infty > 1$$

$\Rightarrow$ The given series is convergent.

Q) Discuss the convergence of

$$1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots$$

Soln, The given series

$$= 1^2 x^0 + 2^2 x^1 + 3^2 x^2 + 4^2 x^3 + \cdots + n^2 x^{n-1} + \cdots$$

Let the general term of the above series be U_n.

$$\therefore U_n = n^2 x^{n-1} \Rightarrow U_{n+1} = (n+1)^2 x^n$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{n^2 x^{n-1}}{(n+1)^2 x^n} = \frac{1}{(1+\frac{1}{n})^2} \times \frac{1}{x}$$

$$\Rightarrow \lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \left[\lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2}\right] \times \frac{1}{x} = \frac{1}{x}$$

If $\frac{1}{x} > 1$ i.e. $x < 1$ then $\sum U_n$ is convergent.

If $\frac{1}{x} < 1$ i.e. $x > 1$ then $\sum U_n$ is divergent.

If $\frac{1}{x} = 1$ / i.e. $x = 1$ then the series becomes

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots$$

$$= \frac{1}{1^{-2}} + \frac{1}{2^{-2}} + \frac{1}{3^{-2}} + \frac{1}{4^{-2}} + \cdots$$

$$= \sum \frac{1}{n^{-2}} \qquad \text{Here } p = -2$$

So, the series is divergent when $x = 1$.

Q Test the convergence of

$$\frac{1}{1^p} + \frac{1}{3^p}x + \frac{1}{5^p}x^2 + \cdots + \frac{x^n}{(2n+1)^p} + \cdots$$

Soln The given series

$$= \frac{1}{1^p}x^0 + \frac{1}{3^p}x + \frac{1}{5^p}x^2 + \cdots + \frac{x^{n-1}}{(2n-1)^p}$$

$$+ \frac{x^n}{(2n+1)^p} + \cdots$$

Let $U_n =$ n^{th} term of the above series

$$\Rightarrow U_n = \frac{x^{n-1}}{(2n-1)^p} \Rightarrow U_{n+1} = \frac{x^n}{(2n+1)^p}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{x^{n-1}}{(2n-1)^p} \times \frac{(2n+1)^p}{x^n}$$

$$= \frac{1}{x} \times \left(\frac{2n+1}{2n-1}\right)^p = \frac{1}{x}\left[\frac{2+\frac{1}{n}}{2-\frac{1}{n}}\right]^p$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{1}{x} \times \underset{n\to\infty}{Lt}\left[\frac{2+\frac{1}{n}}{2-\frac{1}{n}}\right]^p$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{1}{x} \times 1^p = \frac{1}{x}$$

If $\frac{1}{x} > 1$ i.e. $x < 1$ then $\sum U_n$ is cgt.

If $\frac{1}{x} < 1$ i.e. $x > 1$ then $\sum U_n$ is dgt.

If $\frac{1}{x} = 1$ then this test fails.

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So, the given series

$$= \frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \cdots$$

$$= \left(\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right) - \left(\frac{1}{2^p} + \frac{1}{4^p} + \cdots\right)$$

$$= \left(\frac{1}{1^p} + \frac{1}{2^p} + \cdots\right) - \frac{1}{2^p}\left(\frac{1}{1^p} + \frac{1}{2^p} + \cdots\right)$$

$$= \left(\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\right)\left(1 - \frac{1}{2^p}\right)$$

$$= \left(1 - \frac{1}{2^p}\right) \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$\because \sum \frac{1}{n^p}$ is cgt if $p > 1$

and it is dgt if $p \leq 1$.

Hence, the series in question is cgt if $p > 1$

and dgt if $p \leq 1$.

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