Quantum Mechanics-Section18

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1 The wave function in momentum space

We have already established the wave function ψ as a function of x and t.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{i(px-Et)/\hbar}$$

We can now discuss the physical significance of $\phi(p)$. It is sufficient to consider $t = 0$, since $\phi(p)$ does not have any time dependence. Therefore,

$$\psi(x) = \int dp\, \phi(p)\, e^{ipx/\hbar}$$

By the help of inverse Fourier transform we find

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \psi(x)\, e^{-ipx/\hbar}$$

Now,

$$\int dp\, \phi^*(p)\phi(p) = \int dp\, \phi^*(p) \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \psi(x)\, e^{-ipx/\hbar} \quad (1)$$

$$= \int dx\, \psi(x) \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi^*(p) e^{-ipx/\hbar} \quad (2)$$

$$= \int dx\, \psi(x)\psi^*(x) = 1 \quad (3)$$

The result is known as Parseval's theorem, it states that if a function is normalized to 1, so is its Fourier Transform.

Consider

$$< p >= \int dx\, \psi^*(x) \hat{p} \psi(x) \quad (4)$$

$$= \int dx\, \psi^*(x) \frac{\hbar}{i} \frac{\partial \psi(x)}{\partial x} \quad (5)$$

$$= \int dx\, \psi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p) e^{ipx/\hbar} \quad (6)$$

$$= \int dp\, \phi(p) p \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \psi^*(x) e^{ipx/\hbar} \quad (7)$$

$$= \int dp\, \phi(p)\, p\, \phi^*(p) \quad (8)$$

This result suggests that $\phi(p)$ should be interpreted as the wave function in momentum space, with $|\phi(p)|^2$ yielding the probability density for finding the particle with momentum p. We may define $\phi(p,t)$ by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p,t) e^{ipx/\hbar}$$

Like the momentum operator $\hat{p} \equiv \frac{\hbar}{i}\frac{\partial}{\partial x}$ in ordinary space, we may also express x as position operator in momentum space

$$< x >= \int dp\, \phi^*(p,t) \left(i\hbar \frac{\partial}{\partial p} \right) \phi(p,t)$$

where

$$\hat{x} \equiv i\hbar \frac{\partial}{\partial p}$$

References

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[1] Figures are collected from online resources.