B.Sc. Part II (Hons). 4th Paper

Vector calculus By Alok kr.

Differentiation of vector

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{(\vec{r}+\delta\vec{r})-\vec{r}}{\delta t}$$

* $$\frac{d}{dt}(\vec{a}+\vec{b}) = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$$

Proof

Let $\vec{a}$ and $\vec{b}$ be two differentiable vectors, which are functions of variable scalar t.

Let $\delta\vec{a}$ = increment in $\vec{a}$ due to increment δt

and $\delta\vec{b}$ = increment in $\vec{b}$ due to increment δt

$$\therefore \frac{d}{dt}(\vec{a}+\vec{b}) = \lim_{\delta t \to 0} \frac{\left[\{(\vec{a}+\delta\vec{a})+(\vec{b}+\delta\vec{b})\} - (\vec{a}+\vec{b})\right]}{\delta t}$$

$$= \lim_{\delta t \to 0} \left(\frac{\delta\vec{a}+\delta\vec{b}}{\delta t}\right)$$

$$= \lim_{\delta t \to 0} \left(\frac{\delta\vec{a}}{\delta t} + \frac{\delta\vec{b}}{\delta t}\right)$$

$$= \lim_{\delta t \to 0} \frac{\delta\vec{a}}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\vec{b}}{\delta t}$$

$$= \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$$

Hence, $$\frac{d}{dt}(\vec{a}+\vec{b}) = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}.$$

* $$\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}$$

Proof. Let $\vec{a}$ and $\vec{b}$ be two differentiable vector functions of the scalar t.

Let $\delta\vec{a}$ and $\delta\vec{b}$ be ~~two~~ the respective increments in $\vec{a}$ and $\vec{b}$ due to increment δt in t.

$$\therefore \delta(\vec{a}\cdot\vec{b}) = (\vec{a}+\delta\vec{a})\cdot(\vec{b}+\delta\vec{b}) - \vec{a}\cdot\vec{b}$$

$$\Rightarrow \frac{\delta(\vec{a}\cdot\vec{b})}{\delta t} = \frac{(\vec{a}+\delta\vec{a})\cdot(\vec{b}+\delta\vec{b}) - \vec{a}\cdot\vec{b}}{\delta t}$$

$$= \frac{\vec{a}\cdot\delta\vec{b} + \delta\vec{a}\cdot\vec{b} + \delta\vec{a}\cdot\delta\vec{b}}{\delta t}$$

$$\Rightarrow \frac{\delta}{\delta t}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{\delta\vec{b}}{\delta t} + \frac{\delta\vec{a}}{\delta t}\cdot\vec{b} + \frac{\delta\vec{a}}{\delta t}\cdot\frac{\delta\vec{b}}{\delta t}\delta t$$

Taking $\lim\limits_{\delta t\to 0}$ both sides, we get

$$\lim_{\delta t\to 0}\frac{\delta}{\delta t}(\vec{a}\cdot\vec{b}) = \lim_{\delta t\to 0}\left(\vec{a}\cdot\frac{\delta\vec{b}}{\delta t} + \frac{\delta\vec{a}}{\delta t}\cdot\vec{b} + \frac{\delta\vec{a}}{\delta t}\cdot\frac{\delta\vec{b}}{\delta t}\delta t\right)$$

$$= \vec{a}\cdot\left(\lim_{\delta t\to 0}\frac{\delta\vec{b}}{\delta t}\right) + \left(\lim_{\delta t\to 0}\frac{\delta\vec{a}}{\delta t}\right)\cdot\vec{b}$$

$$+ \lim_{\delta t\to 0}\left(\frac{\delta\vec{a}}{\delta t}\right)\cdot\lim_{\delta t\to 0}\left(\frac{\delta\vec{b}}{\delta t}\right)\lim_{\delta t\to 0}\delta t$$

$$\Rightarrow \frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{b} + \frac{d\vec{a}}{dt}\cdot\frac{d\vec{b}}{dt}\times 0$$

$$\Rightarrow \frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}$$ Proved

$*$ $$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$$

Proof

Let $\vec{a}$ and $\vec{b}$ be two differentiable vector functions of scalar t.

Let $\delta\vec{a}$ = small increment in $\vec{a}$ due to increment δt

$\delta\vec{b}$ = small increment in $\vec{b}$ due to increment δt

$\therefore$ increment in $\vec{a}\times\vec{b}$ i.e. $\delta(\vec{a}\times\vec{b})$

$$= (\vec{a}+\delta\vec{a})\times(\vec{b}+\delta\vec{b}) - \vec{a}\times\vec{b}$$

$$= \vec{a}\times\delta\vec{b} + \delta\vec{a}\times\vec{b} + \delta\vec{a}\times\delta\vec{b}$$

$$\Rightarrow \delta(\vec{a}\times\vec{b}) = \delta\vec{a}\times\vec{b} + \vec{a}\times\delta\vec{b} + \delta\vec{a}\times\delta\vec{b}$$

$$\Rightarrow \frac{\delta(\vec{a}\times\vec{b})}{\delta t} = \frac{\delta\vec{a}}{\delta t}\times\vec{b} + \vec{a}\times\frac{\delta\vec{b}}{\delta t} + \frac{\delta\vec{a}}{\delta t}\times\delta\vec{b}$$

$$\Rightarrow \frac{\delta}{\delta t}(\vec{a}\times\vec{b}) = \frac{\delta\vec{a}}{\delta t}\times\vec{b} + \vec{a}\times\frac{\delta\vec{b}}{\delta t} + \frac{\delta\vec{a}}{\delta t}\times\frac{\delta\vec{b}}{\delta t}\cdot\delta t$$

Taking $\lim_{\delta t\to 0}$ both sides, we get

$$\Rightarrow \frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt} + 0$$

Hence, $$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$$

* Also, $\frac{d}{dt}\left(\vec{a} \times \frac{d\vec{a}}{dt}\right) = \vec{a} \times \frac{d^2\vec{a}}{dt^2}$

Proof:

$$LHS = \frac{d}{dt}\left(\vec{a} \times \frac{d\vec{a}}{dt}\right)$$

$$= \frac{d\vec{a}}{dt} \times \frac{d\vec{a}}{dt} + \vec{a} \times \frac{d}{dt}\left(\frac{d\vec{a}}{dt}\right)$$

$$= \vec{0} + \vec{a} \times \frac{d^2\vec{a}}{dt^2}$$

$$= \vec{a} \times \frac{d^2\vec{a}}{dt^2} \quad \underline{\text{Proved}}$$

* If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\vec{k}$ then find $\left|\frac{d^2\vec{r}}{dt^2}\right|$.

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{d}{dt}(\cos t)\,a\vec{i} + \frac{d}{dt}(\sin t)\,a\vec{j} + \frac{d}{dt}(t)\,\vec{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = -\sin t\,a\vec{i} + \cos t\,a\vec{j} + \vec{k}$$

Again, differentiating w.r.to t, we get

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -\cos t\,a\vec{i} - \sin t\,a\vec{j}$$

$$= -a(\cos t\,\vec{i} + \sin t\,\vec{j})$$

$$\Rightarrow \left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2}$$

$$= \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \underline{\underline{a}}.$$