B. Sc. Part II (Hons.)

ALOK KUMAR
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Differential Equations
(4th Paper)

ORTHOGONAL TRAJECTORIES

If a curve intersects every member of a given family of curves at 90°, it is called orthogonal trajectory. If the angle is not 90°, it is called an oblique trajectory.

Methodology

Cartesian eqn.

1. Differentiate w.r. to x and find $\frac{dy}{dx}$.
2. Eliminate the parameter
3. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.
4. Integrate. You get the solution

I. Find the orthogonal trajectories of family of curves $y = ax^2$, a being a parameter.

Soln

Given, $y = ax^2$ ——(1)

$\Rightarrow \frac{dy}{dx} = 2ax = 2x \times a$

$= 2x \times \frac{y}{x^2}$ $\left[\because y = ax^2 \text{ from (1)}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above equation.

$\Rightarrow -\frac{dx}{dy} = \frac{2y}{x} \Rightarrow x\,dx = -2y\,dy$

$\Rightarrow x\,dx + 2y\,dy = 0$

Integrating, we get

$\frac{x^2}{2} + y^2 = \frac{c}{2}$

$\Rightarrow x^2 + 2y^2 = c$

This is the required orthogonal trajectories of the given eqn.

2. Find the orthogonal trajectories of family of curves
$$3xy = x^3 - a^3,$$ a being parameter of the family.

Soln. Given $3xy = x^3 - a^3$

Diff w.r.to x, we get

$3\left(x\frac{dy}{dx} + y\right) = 3x^2$

$\Rightarrow x\frac{dy}{dx} + y = x^2$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$-x\frac{dx}{dy} + y = x^2$

$\Rightarrow x\frac{dx}{dy} - y = -x^2$

$\Rightarrow x\frac{dx}{dy} + x^2 = y$

Put $x^2 = z$

$\Rightarrow 2x\frac{dx}{dy} = \frac{dz}{dy}$

$\Rightarrow \frac{1}{2}\frac{dz}{dy} + z = y$

$\Rightarrow \frac{dz}{dy} + 2z = 2y$ which is a linear diff. eqn.

$I.F = e^{\int P\,dy} = e^{\int 2\,dy} = e^{2y}$

∴ Its Soln. is given by

$$z \times IF = \int Q \times IF \, dy$$

$$\Rightarrow z e^{2y} = \int 2y e^{2y} dy + k$$

$$\Rightarrow z e^{2y} = 2y \int e^{2y} dy - 2\int \left[\frac{dy}{dy} \int e^{2y} dy\right] dy$$

$$= \cancel{2}y \cdot \frac{e^{2y}}{\cancel{2}} - \cancel{2} \times \int \frac{e^{2y}}{\cancel{2}} + k$$

$$\Rightarrow z e^{2y} = y e^{2y} - \frac{e^{2y}}{2} + k$$

$$\Rightarrow z = y - \frac{1}{2} + k e^{-2y}$$

This is the ~~required~~ orthogonal trajectory

Q Find the orthogonal trajectory of $x^2+y^2 = 2ax$.

Soln

$$\because x^2+y^2 = 2ax \quad \text{---(1)}$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 2a$$

$$\Rightarrow x + y \frac{dy}{dx} = a \quad \text{---(2)}$$

From (1) $a = \dfrac{x^2+y^2}{2x}$

so (2) becomes

$$x + y\frac{dy}{dx} = \frac{x^2+y^2}{2x}.$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above eqn.

$$\Rightarrow \quad x - y\frac{dx}{dy} = \frac{x^2+y^2}{2x}$$

$$\Rightarrow \quad x - \frac{x^2+y^2}{2x} = y\frac{dx}{dy}$$

$$\Rightarrow \frac{x^2-y^2}{2x} = y\frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

which is a homogeneous eqn.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore\ v + x\frac{dv}{dx} = \frac{2x \cdot vx}{x^2 - v^2x^2} = \frac{2v}{1-v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{v+v^3}{1-v^2} = \frac{v(1+v^2)}{1-v^2}$$

$$\Rightarrow \quad \frac{(1-v^2)}{v(1+v^2)}\,dv = \frac{dx}{x} \Rightarrow \frac{dx}{x} = \frac{dv}{v} - \frac{2v\,dv}{1+v^2}$$

integrating, $\Rightarrow \log x = \log v - \log(1+v^2) + \log k$

$$\Rightarrow x(1+v^2) = vk \Rightarrow x\left(1+\frac{y^2}{x^2}\right) = \frac{y}{x}k$$

$$\Rightarrow \frac{1}{x}(x^2+y^2) = \frac{y}{x}k \Rightarrow x^2+y^2 = ky$$

This is the required o.t.