Mathematical Physics.

B.Sc. Physics Part-III

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In last note we have discussed gradient. For any scalar function $\phi(x,y,z)$, the grad ϕ or $\vec{\nabla}\phi$ is defined as

$$\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

↑ vector differential operator

one can use $\hat{i}, \hat{j}, \hat{k}$ in place of $\hat{x}, \hat{y}, \hat{z}$

(A). Divergence: If we operate ∇ on a vector ~~$\vec{A}$~~ $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$. We dot $\vec{\nabla}$ into the $\vec{A}$ and obtain

$$\boxed{\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

$\nabla\cdot\vec{A} \rightarrow$ divergence of $\vec{A}$.
Scalar quantity

Some Examples:

(a) Evaluating $\nabla\cdot\vec{r}$, $\quad \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$$\nabla\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\boxed{\nabla\cdot\vec{r} = 3}$$

(b) ~~Calculate $\nabla^2 f(r)$~~ Prove $\nabla\cdot\nabla\phi = \nabla^2\phi$
where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

$$\nabla\cdot\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi,$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We call it Laplacian operator.

H.W. Prove $\nabla^2\left(\frac{1}{r}\right) = 0$

Hint: $r = \sqrt{x^2+y^2+z^2}$

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

Q Evaluate $\nabla\cdot(\vec{r}\,\phi(|\vec{r}|))$; $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial}{\partial x}[x\,\phi(|\vec{r}|)] + \frac{\partial}{\partial y}[y\,\phi(|\vec{r}|)] + \frac{\partial}{\partial z}[z\,\phi(|\vec{r}|)]$$

$$= \phi(|\vec{r}|) + x\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \phi(r) + y\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} + z\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial z} + \phi(r$$

$$= 3\phi(r) + x\frac{\partial\phi}{\partial r}\left(\frac{x}{r}\right) + y\frac{\partial\phi}{\partial r}\left(\frac{y}{r}\right) + z\frac{\partial\phi}{\partial r}\left(\frac{z}{r}\right)$$

$$= 3\phi(r) + \frac{x^2}{r}\frac{\partial \phi}{\partial r} + \frac{y^2}{r}\frac{\partial \phi}{\partial r} + \frac{z^2}{r}\frac{\partial \phi}{\partial r} = 3\phi(r) + \left(\frac{x^2}{r}+\frac{y^2}{r}+\frac{z^2}{r}\right)\frac{\partial \phi}{\partial r}$$

$$= 3\phi(r) + \frac{r^2}{r}\frac{\partial \phi}{\partial r}$$

$$\boxed{\nabla\cdot[\vec{r}\phi(r)] = 3\phi(r) + r\frac{\partial \phi}{\partial r}}$$

(d) Evaluate $\nabla\cdot(\vec{r}\, r^{n-1})$

Using the above result; we can write

$$\nabla\cdot(\vec{r}\, r^{n-1}) = 3r^{n-1} + r\frac{\partial (r^{n-1})}{\partial r} = 3r^{n-1} + r\cdot(n-1)r^{n-2}$$

$$= 3r^{n-1} + (n-1)r^{n-1}$$

$$= (n+2)r^{n-1}$$

(e) Prove $\int f(\vec{r})\, \nabla\cdot\vec{A}(\vec{r})\, d\vec{x} = -\int \vec{A}\cdot\nabla f\, d\vec{x}$, $d\vec{x} = dx\,dy\,dz$.

$\vec{A}$ or f both vanish at infinity.

Soln.

$$\int_{x,y,z} f(\vec{r})\, \nabla\cdot A(\vec{r}) d\vec{x} = \int_{x,y,z} f\left(\frac{\partial A_x}{\partial x} dx\,dy\,dz + \frac{\partial A_y}{\partial y} dy\,dx\,dz + \frac{\partial A_z}{\partial z} dz\,dx\,dy\right)$$

Now we ~~apply int~~ perform integration by parts.

Taking first term on r.h.s. of above equation.

$$\int_{x,y,z} f\frac{\partial A_x}{\partial x} dx\,dy\,dz = \iint\left[\cancel{[A_x f]_{x=-\infty}^{+\infty}}^{0} - \int f\frac{\partial A_x}{\partial x} dx\right] dy\,dz$$

$$= -\iiint f\frac{\partial A_x}{\partial x} dx\,dy\,dz = -\int_{x,y,z} f\frac{\partial A_x}{\partial x} dx\,dy\,dz$$

Same for other two terms

We can write,

$$\int f(\vec{r})\,\nabla\cdot\vec{A}(\vec{r})\,d\vec{x} = -\int\left(A_x\frac{\partial f}{\partial x}dx\,dy\,dz + A_y\frac{\partial f}{\partial y}dy\,dx\,dz + A_z\frac{\partial f}{\partial z}dz\,dx\,dy\right)$$

or
$$\boxed{\int f(\vec{r})\,\nabla\cdot\vec{A}(\vec{r})\,d\vec{x} = -\int \vec{A}\cdot\nabla f\,d\vec{x}}$$

(f) Prove $\nabla\cdot(\phi\vec{A}) = (\nabla\phi)\cdot\vec{A} + \phi(\nabla\cdot\vec{A})$

(B.) Curl :

curl of any vector $\vec{A}$ is defined by

$$\nabla\times\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$\nabla\times\vec{A} \rightarrow$ Curl of $\vec{A}$

$\rightarrow$ defined as an operator

Determinant must be expanded such that we get derivatives as in this term.

↳ differential operator should operate on A_x, A_y, A_z.

(i) Prove $\nabla \times (\nabla \phi) = 0$

curl grad $\phi = 0$.

$$\nabla \times \nabla \phi = \nabla \times \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial y}\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]$$

$$= \hat{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right] + \hat{j}\left[\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$+ \hat{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

Assuming that ϕ has continuous second partial derivative,

so that $\frac{\partial^2 \phi}{\partial x_i \partial x_j} \equiv \frac{\partial^2 \phi}{\partial x_j \partial x_i}$

$$\boxed{\nabla \times \nabla \phi = 0}$$

<u>H.W.</u> Prove that $\nabla \cdot (\nabla \times \vec{A}) = 0$.

div curl $\vec{A} = 0$.