M. Sc. Semester I (Mathematics)

1st Paper (MAT CC 01)

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Field Extension

Let F be a field. A field K is said to be an extension of F if F is a subfield of K, i.e. $F \subset K$.

$\because$ F is a sub field of a field $K \Rightarrow K(F)$ is a vector space.

* **Degree of a field extension**

The dimension of the vector space $K(F)$ is called degree of K over F. It is denoted by $[K : F]$.

* If the vector space $K(F)$ is finite dimensional, i.e. the degree of K over F is finite then K is said to be a finite extension of F.

EXAMPLES

1. Let F be a field.

$\because$ $F \subset F$

$\Rightarrow$ F is an extension of the field F.

Also, $\{1\}$ is the basis of the vector-space $F(F)$.

$\Rightarrow$ $\dim F(F) = 1$

$\Rightarrow$ $F(F)$ is finite dimensional.

$\Rightarrow$ F is a finite field extension of F.

2. Field R of all real numbers is a subfield of field C of all complex numbers.

$\Rightarrow$ C is an extension of R.

Also, $(1, i)$ is a basis of $C(R)$.

$\Rightarrow$ $\dim C(R) = 2$

$\Rightarrow$ $[C : R] = 2$ ie finite.

$\Rightarrow$ C is a finite extension of R.

* Let Q = field of all rational numbers

Let $Q(\sqrt{2}) = \{a + b\sqrt{2} : \forall a, b \in Q\}$.

$\Rightarrow$ $Q(\sqrt{2})$ is a field and Q is its subfield.

$\Rightarrow$ $Q(\sqrt{2})$ is an extension of Q.

Also, $(1, \sqrt{2})$ is a basis of the vector space $Q(\sqrt{2})$ over the field Q.

$\Rightarrow$ $[Q(\sqrt{2}) : Q] = 2$

ie degree of $Q(\sqrt{2})$ over Q is finite.

$\Rightarrow$ $Q(\sqrt{2})$ is a finite extension of Q.

* Let Q = field of all ~~real~~ rational numbers.

Let $Q(\sqrt{2}, \sqrt{3}) = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} : a, b, c, d \in Q\}$

$\Rightarrow$ $Q(\sqrt{2}, \sqrt{3})$ is a field and Q is its subfield.

Also, $\{1, \sqrt{2}, \sqrt{3}, \sqrt{2}\sqrt{3}\}$ is a basis for $Q(\sqrt{2}, \sqrt{3})$.

$\Rightarrow$ $[Q(\sqrt{2}, \sqrt{3}) : Q] = 4$.

$\Rightarrow \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a finite field extension of $\mathbb{Q}$.

Simple Field Extension

Let F be a field and $E \subseteq F$. Then the smallest subfield of F containing E is said to be generated by E.

Let E be an extension of a field F and $S \subseteq E$. Then the subfield of E generated by $F \cup S$ is called subfield of E generated by S over F and is denoted by $F(S)$.