M.Sc. Sem II Paper XI (CC-11)

FUNCTIONAL ANALYSIS

NORM Let X be a non-empty set

In each space there is defined a notion of the distance from an arbitrary element to the origin, i.e. a notion of the size of an arbitrary element.

The size of an element x is a real number denoted as $\|x\|$ and is called its NORM.

It has following properties

1. $\|x\| \geq 0$
2. $\|x\| = 0 \Leftrightarrow x = 0$
3. $\|-x\| = \|x\|$
4. $\|x+y\| \leq \|x\| + \|y\|$

* The set of all bounded continuous real functions defined on the closed unit interval has a metric, defined below.

$$\|f\| = \sup\{|f(x)| : x \in [0,1]\}$$

which is briefly written as

$$\|f\| = \sup |f(x)|$$

and $d(f,g) = \|f-g\| = \sup |f(x) - g(x)|$

This metric space is denoted as $\mathcal{C}[0,1]$.

* ~~Ab~~ <u>Linear space</u>

Let L be a non-empty set.

Let each pair of elements x, y of L can be combined by a process called addition to yield an element z in L, denoted as

$$z = x + y.$$

Also, let $x + y = y + x$ and $x + (y+z) = (x+y) + z$.

Let, there exists in L, a unique element denoted by 0, (called zero element or origin) such that $x + 0 = x \;\forall\; x \in L$

Again, let to each element $x \in L$, $\exists$ a unique element $-x$ (negative of x) s.t.

$$x + (-x) = 0$$

Also, let the system of real numbers or complex numbers as scalars.

Let, a scalar α and an element $x \in L$ can be combined by a process called <u>SCALAR MULTIPLICATION</u> to yield an element y in L denoted as

$y = \alpha x$ such that

(a) $\alpha(x+y) = \alpha x + \alpha y$

(b) $(\alpha+\beta)x = \alpha x + \beta x$

(c) $(\alpha\beta)x = \alpha(\beta x)$

(d) $1 \cdot x = x$.

The algebraic system L defined by these operations and axioms is called a <u>linear space</u>.

<u>Real linear space</u> : when scalars are real numbers only.

<u>Complex linear space</u> : when scalars are complex numbers only.

Normed linear space It is a linear space on which there is defined a norm, i.e., a function which assigns to each element x in the ~~s~~ space a real number $\|x\|$ in such a manner that

(a) $\|x\| \geq 0$

(b) $\|x\| = 0 \Leftrightarrow x = 0$

(c) $\|x+y\| \leq \|x\| + \|y\|$

(d) $\|\alpha x\| = |\alpha| \, \|x\|$.

Here, element x is a vector and α is a scalar.