M.Sc. Sem I

Paper I (CC 01)

Unit 3

Types of Extension:

Let E be an extension field of a field F. Let $a \in E$. Then a is called <u>ALGEBRAIC</u> ~~[illegible]~~ over F if a is the zero of some non-zero polynomial in $F[x]$.

If a is not algebraic over F, it is called <u>transcendental</u> over F.

An extension E of F is called <u>algebraic extension</u> of F if every element of E is algebraic over F.

If E is not an algebraic extension of F, it is called a <u>transcendental extension</u> of F.

An extension of F of the form $F(a)$ is called a simple extension of F.

Theorem (TRANSITIVITY of finite exte.)

Statement: If L is a finite extension of K and K is a finite extension of F, then L is a finite extension of F.

Also, $[L:F] = [L:K][K:F]$

Proof

Let K be a subfield of L and F be a subfield of K.

Let $[K:F] = m$, $[L:K] = n$.

Let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $S_2 = \{\beta_1, \beta_2, \ldots \beta_n\}$ be the bases of vector spaces $K(F)$ and $L(K)$ respectively.

We shall prove that the set $S = \{\alpha_i \beta_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of the vector space $L(F)$.

Now, for any scalars a_{ij} ($i = 1$ to m, $j = 1$ to n), we have

$$\sum_{i,j} a_{ij} (\alpha_i \beta_j) = 0$$

$$\Rightarrow \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \alpha_i \beta_j = 0$$

$$\Rightarrow \sum_{j=1}^{n} \left[\sum_{i=1}^{m} a_{ij} \alpha_i \right] \beta_j = 0$$

$$\Rightarrow \sum_{i=1}^{m} a_{ij} \alpha_i = 0 \quad \forall\, j = 1, 2, \dots n$$

$\left[\because S_2 \text{ is linearly independent and } \sum a_{ij} \alpha_i \in K \ \forall j \right]$

$$\Rightarrow \sum_{i=1}^{m} a_{i1} \alpha_i = 0, \ \sum_{i=1}^{m} a_{i2} \alpha_i = 0, \dots$$

$$\sum_{i=1}^{m} a_{in} \alpha_i = 0.$$

$\Rightarrow$ But S_1 is linearly independent

$$\Rightarrow a_{i1} = 0, \ a_{i2} = 0, \ \dots \ a_{in} = 0 \ \forall i$$

$$\Rightarrow a_{ij} = 0 \ \forall\, i, j$$

$\Rightarrow S$ is a linearly independent subset of L. ~~Then~~

Now, let $v \in L$ be arbitrary.

$\because S_2$ is a basis of $L(K)$

$\Rightarrow \exists$ elements $v_1, v_2, \dots, v_n$ in K

such that $\quad v = \sum_{j=1}^{n} v_j \beta_j$

Also, each $\gamma_j \in K$ and S_1 being a basis of $K(F)$, there exist elements $a_{1j}, a_{2j}, a_{3j}, \ldots a_{mj}$ in F such that

$$\gamma_j = \sum_{i=1}^{m} a_{ij} \alpha_i \quad \forall j$$

$$\Rightarrow \gamma = \sum_{j=1}^{n} \gamma_j \beta_j = \sum_{j=1}^{n} \left[\sum_{i=1}^{m} a_{ij} \alpha_i \right] \beta_j$$

$$\Rightarrow \gamma = \sum_{i,j} a_{ij} (\alpha_i \beta_j)$$

Thus, each element of L is expressible as a linear combination of elements of S. ~~and~~

$\Rightarrow S$ generates L

$\Rightarrow S$ is a basis of $L(F)$.

Thus, $\dim L(F) = mn$

i.e. $[L : F] = mn = nm$

$$\Rightarrow [L : F] = n \times m = [L : K][K : F]$$

Proved.