Mathematical Physics
B.Sc Physics Part-III
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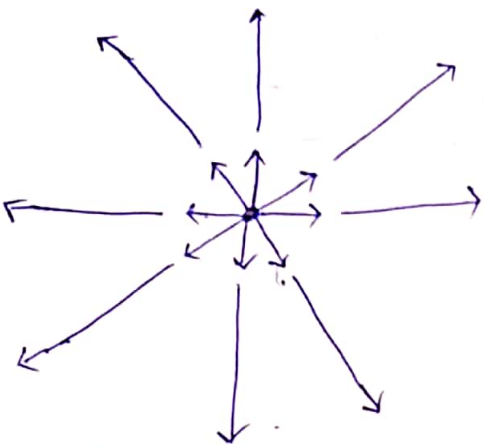
Date - 07-07-2020

Geometrical Interpretation of divergence and Curl:-

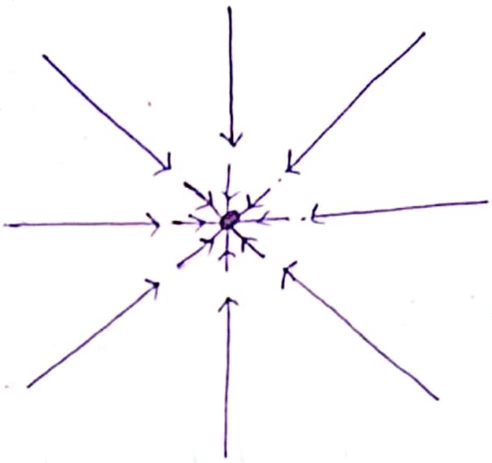
Divergence — $\cancel{\boxed{\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}}$

$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$, where $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$

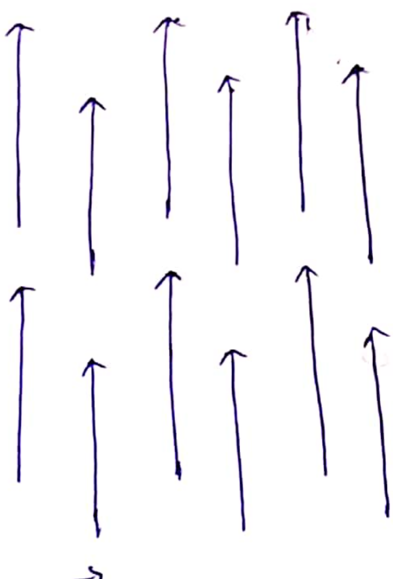
$\nabla\cdot\vec{A} \rightarrow$ Measure of how much the vector $\vec{A}$ diverges or spreads out from the particular point in the Question



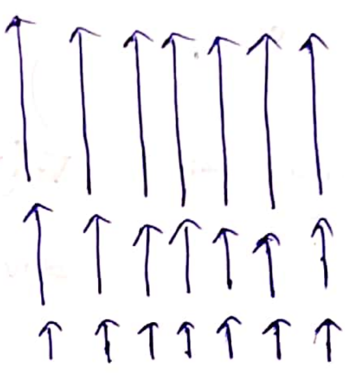
$\nabla\cdot\vec{A} = +ive$



$\nabla\cdot\vec{A} = -ive$



$\nabla\cdot\vec{A} = 0$

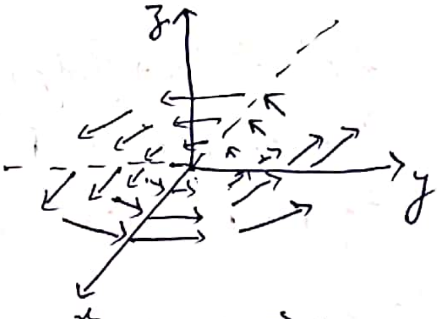


$\nabla\cdot\vec{A} = +ive$

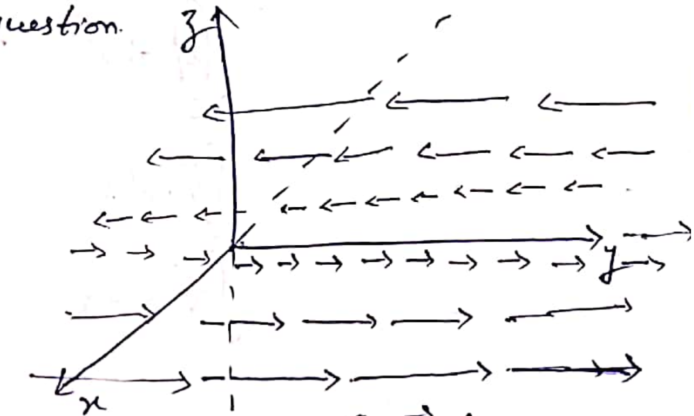
Curl: $\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$\nabla \times \vec{A} \rightarrow$ measure of how much vector $\vec{A}$ curls around the particular point in question.



$\nabla \times \vec{A} \neq 0$



$\nabla \times \vec{A} \neq 0$

In Figures on page (1) $\nabla \times \vec{A} = 0$.

Some Identities:

(i) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(ii) $\nabla(\vec{A}\cdot\vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$

(iii) $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$

(iv) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

(v) $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f)$

(vi) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$

(vii) $\nabla\cdot(\nabla\times\vec{A}) = 0$

(viii) $\nabla\times(\nabla f) = 0$

(ix) $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2 A$

H.W. Take vectors $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ & $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ and verify above identities.

$A_x, A_y, A_z \,/\, B_x, B_y, B_z$ all are functions of (x, y, z)

Vector Integration:

How to integrate vectors → Reduce vector integral to scalar integral.

(i) Line Integrals – Three type of line integral.

$$\int_C \phi \, d\vec{r} \quad — (1)$$

$$\int_C \vec{V} \cdot d\vec{r} \quad — (2) \qquad ; \; d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

length element

$$\int_C \vec{V} \times d\vec{r} \quad — (3)$$

Integral is over Contour C

Contour C → May be open or closed (loop)

→ ϕ is scalar, $\vec{V}$ is vector.

→ The second integral is most important of the three

→ ~~To evalute the~~ consider the second integral

$$\int_C \vec{V} \cdot d\vec{r}.$$

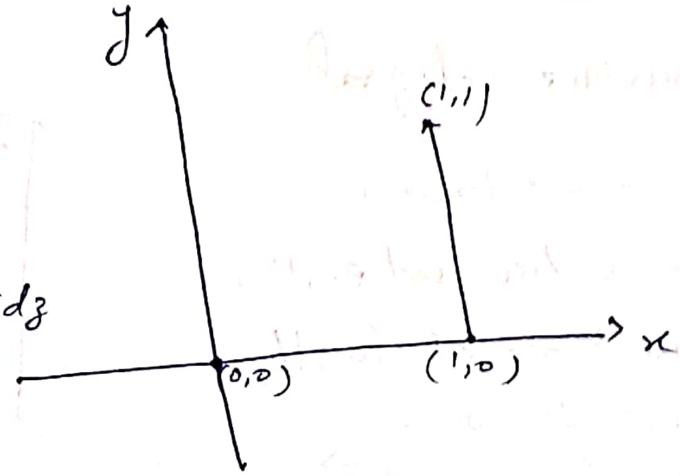
If we want to calculate work done by a force $\vec{F}$ which varies along the path.

$$W = \int \vec{F} \cdot d\vec{r}$$

. Consider the force applied on a body is $\vec{F} = -y^2\hat{x} + x\hat{y}$

We want to calculate work done from $(0,0)$ to $(1,1)$.

$$W = \int_{0,0}^{1,1} \vec{F}\cdot d\vec{r}$$



$$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

Q —

$$\vec{F}\cdot d\vec{r} = (-y^2\hat{x} + x\hat{y})\cdot(\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz)$$

$$= -y^2\,dx + x\,dy$$

$$W = \int_{0,0}^{1,1} (-y^2\,dx + x\,dy)$$

limit for x is ranging from (0,1) and for y it is same (0,1).
So we separate both the integrals corresponding to x & y.

$$W = -\int_0^1 y^2\,dx + \int_0^1 x\,dy$$

To evaluate first integral we need to know what is y for $x \rightarrow (0,1)$. See Fig. for x going from 0 to 1 y is 0

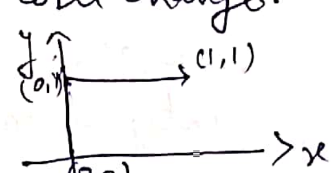
$$W = -\int_0^1 0\,dx + \int_0^1 x\,dy = \int_0^1 x\,dy$$

Now we need to calculate what is x for y going from 0 to 1.

In this case x is 1.

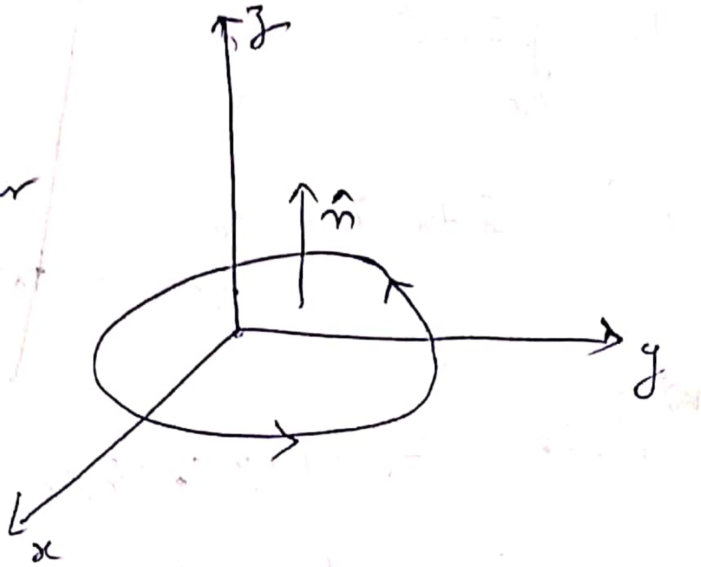
$$\boxed{W = \int_0^1 1\,dy = 1.}$$

Note: changing the path. Work done W will change.
So W is path dependent.
See figure. End points are same but path is changed. Hence W will change.



(ii) Surface integrals.

Here we have same form as discussed earlier but in place of length element we have area element $d\vec{\sigma}$



$d\vec{\sigma} \rightarrow$ vector

$d\vec{\sigma} = \hat{n}\, dA$, $\hat{n}$ - unit normal, for Fig above it is indicating positive normal

The three types are

$$\int_S \phi\, d\vec{\sigma}$$

$$\int_S \vec{V}\cdot d\vec{\sigma} \rightarrow \text{most encountered.}$$

$$\int_S \vec{V}\times d\vec{\sigma}$$

$\int_S \vec{V}\cdot d\vec{\sigma}$ — denotes flow or flux through the given surface

(iii) Volume Integrals:

For volume element $d\tau$ we can write volume ~~element~~ integral

$$\int_V \vec{V} d\tau = \hat{x}\int_V V_x d\tau + \hat{y}\int_V V_y d\tau + \hat{z}\int_V V_z d\tau$$

$d\tau = dx\,dy\,dz \rightarrow$ scalar quantity

We will discuss more examples and elaborate the vector integration in next note.