B. Sc. Part II (Hons) 4^{th} Paper

DYNAMICS

velocity $= \frac{dx}{dt} = \dot{x}$

acceleration $= \frac{dv}{dt} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$

1. The coordinates of a moving point at time t are given by

$x = a(2t + \sin 2t)$, $y = a(1 - \cos 2t)$.

Prove that its acceleration is constant and find the direction of motion at time t

Soln Given that

$$x = a(2t + \sin 2t)$$

$$\Rightarrow \frac{dx}{dt} = a(2 + 2\cos 2t)$$

$$\Rightarrow \frac{d^2x}{dt^2} = a \times 2 \times (-2)\sin 2t = -4a\sin 2t$$

Again

$$y = a(1 - \cos 2t)$$

$$\Rightarrow \frac{dy}{dt} = a \times 2\sin 2t$$

$$\Rightarrow \frac{d^2y}{dt^2} = a \times 2 \times 2\cos 2t = 4a\cos 2t$$

Now. acceleration $= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$

$= \sqrt{(-4a\sin 2t)^2 + (4a\cos 2t)^2} = 4a$

$=$ constant proved (1st Part)

Now, direction of motion $= \tan^{-1}\frac{\dot{y}}{\dot{x}}$

$$= \tan^{-1}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \tan^{-1}\left(\frac{2a\sin 2t}{a(2+2\cos 2t)}\right)$$

$$= \tan^{-1}\left(\frac{\sin 2t}{1+\cos 2t}\right)$$

$$= \tan^{-1}\left(\frac{\cancel{2}\sin t.\cancel{\cos t}}{\cancel{2}\cos^{\cancel{2}} t}\right)$$

$$= \tan^{-1}(\tan t) = t$$

$\Rightarrow$ Direction makes an angle t with x-axis

Q. A particle is acted on by a force parallel to y-axis whose acceleration (always directed towards x-axis) is μy^{-2} and when $y = a$, it is projected parallel to x-axis with velocity $\sqrt{\frac{2\mu}{a}}$. Prove that it will describe a cycloid.

Soln Given that

acceleration directed towards x-axis $= \frac{\mu}{y^2}$

$\Rightarrow$ i.e. y is decreasing

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{\mu}{y^2} \quad \text{[(-) sign shows that } y \text{ is decreasing]}$$

$$\Rightarrow v\frac{dv}{dy} = -\frac{\mu}{y^2}$$

$$\Rightarrow \quad v\,dv = -\frac{\mu}{y^2}\,dy$$

Integrating, we get

$$\frac{v^2}{2} = \frac{\mu}{y} + c \quad \text{——— (1)}$$

Initially (i.e. at $y=a$, $v=0$) $\Rightarrow$ (1) $\Rightarrow 0 = \frac{\mu}{a} + c$

$\Rightarrow c = -\frac{\mu}{a}$. Putting this value in (1), we have

$$\therefore (1) \Rightarrow \frac{v^2}{2} = \frac{\mu}{y} - \frac{\mu}{a}$$

$$\Rightarrow v^2 = 2\mu\left(\frac{1}{y} - \frac{1}{a}\right) \quad \text{——— (2)}$$

Initially $y = a$ $\Rightarrow$ velocity is parallel to x-axis

$\because$ Particle is acted only by a force $\parallel$ to y-axis

$\Rightarrow \frac{dy}{dt}$ = velocity parallel to y-axis in the sense of y-increasing

$$\Rightarrow v = \frac{dy}{dt} = -\sqrt{2\mu\left(\frac{1}{y} - \frac{1}{a}\right)}$$ [(−) sign as y is decreasing] [using (2)]

$$\Rightarrow \frac{dy}{dt} = -\sqrt{\frac{2\mu}{a}}\sqrt{\frac{a-y}{y}} \quad \text{——— (3)}$$

Given that accn is always directed towards x-axis.

$\Rightarrow$ accn parallel to x-axis $= 0$

$$\Rightarrow \frac{d^2x}{dt^2} = 0 \Rightarrow \frac{dx}{dt} = \text{constant}.$$

But, given that

initially velocity parallel to x-axis $= \sqrt{\dfrac{2\mu}{a}}$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{2\mu}{a}} \quad \text{— (4)}$$

from (3) and (4)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\sqrt{\frac{a-y}{y}}$$

$$\Rightarrow \sqrt{\frac{y}{a-y}}\, dy + dx = 0 \quad \text{— (5)}$$

put $y = a\cos^2\theta \Rightarrow dy = -2a\cos\theta.\sin\theta\, d\theta$

and $\sqrt{\dfrac{y}{a-y}} = \sqrt{\dfrac{a\cos^2\theta}{a\sin^2\theta}} = \cot\theta$

(5) $\Rightarrow$ $\cot\theta.(-)2a\cos\theta\sin\theta\, d\theta + dx = 0$

integrating

$$\Rightarrow -2a\int \frac{\cos\theta}{\cancel{\sin\theta}}.\cos\theta\,\cancel{\sin\theta}\, d\theta + \int dx = 0$$

$$\Rightarrow x = a\int (1+\cos 2\theta)\, d\theta + K$$

$$\Rightarrow x = a\left(\theta + \frac{\sin 2\theta}{2}\right) + K \quad \text{— (6)}$$

Initially, when $x = 0$, $y = a$.

$\because$ $y = a\cos^2\theta$ (we had supposition) $\Rightarrow a = a\cos^2\theta \Rightarrow \theta = 0$.

(6) $\Rightarrow$ $0 = a(0+0) + K \Rightarrow K = 0$. Put $K=0$ in (6)

$\therefore$ $x = a\left(\theta + \dfrac{\sin 2\theta}{2}\right) \Rightarrow x = \dfrac{a}{2}(2\theta + \sin 2\theta)$ — (7)

(7) and $y = a\cos^2\theta = \dfrac{a}{2}(1+\cos 2\theta)$ Put $\dfrac{a}{2} = A$, $2\theta = t$

$\Rightarrow$ $x = A(t + \sin t)$, $y = A(1+\cos t)$

cycloid equation