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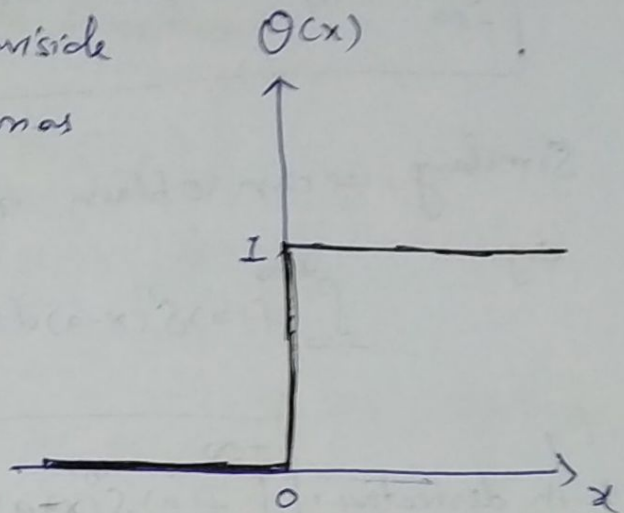
(Classical Electrostatics - 03)

Delta function continued--

Derivative of the Delta function!

First, we define the ~~How~~ Heaviside function or the step function as

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}$$

The derivative of step function $\Theta(x)$ gives the delta function

$$\frac{d\Theta(x)}{dx} = \delta(x)$$

For derivative of delta function we use Fourier transform and write $\frac{d}{dx}(\delta(x))$ as.

$$\frac{d\delta(x)}{dx} = \delta'(x) = \frac{1}{2\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} e^{ikx} dk$$

$$\text{Or } \delta'(x) = \frac{d\delta(x)}{dx} = \frac{i}{2\pi} \int_{-\infty}^{+\infty} k e^{ikx} dk$$

Next, we can also obtain derivative in another way by doing integration by parts of $\delta'(x-a)$.

$$\int_{-\infty}^{+\infty} f(x) \delta'(x-a) dx = f(x) \delta(x-a) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f'(x) \delta(x-a) dx$$

$$= 0 - f'(a)$$

$$\boxed{\int_{-\infty}^{+\infty} f(x) \delta'(x-a) dx = -f'(a)}$$

Similarly, we can obtain n -th derivative of δ -function

e.g.

$$\int_{-\infty}^{+\infty} f(x) \delta''(x-a) dx = (-1)^2 f''(a) = f''(a)$$

$$n\text{-th derivative: } \boxed{\int_{-\infty}^{+\infty} f(x) \delta^{(n)}(x-a) dx = (-1)^n f^{(n)}(a)}$$

$$\text{Here } \delta^{(n)}(x-a) = \frac{d^n}{dx^n} (\delta(x-a))$$

$$f^{(n)}(a) = \frac{d^n [f(x)]}{dx^n} \Big|_{x=a}$$

From n -th derivative result, if we put $f(x) = 1$, $n=1$,

we obtain

$$\int_{-\infty}^{+\infty} \delta'(x-a) dx = 0$$

Some more properties of derivative of delta function:

$$(i) \quad \delta'(x) = -\delta'(-x)$$

$$(ii) \quad x \delta'(x) = -\delta(x)$$

$$(iii) \quad x^2 \delta'(x) = 0$$

$$(iv) \quad x^2 \delta''(x) = 2\delta(x). \quad \text{etc.}$$

Similarly one can obtain expressions for higher derivatives

Three-dimensional delta function:

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z),$$

$$\text{where } \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\int \delta^3(\vec{r}) d\tau = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \delta(x)\delta(y)\delta(z) = 1$$

$$\text{And } \int \delta^3(\vec{r}-\vec{a}) f(\vec{r}) d\tau = f(\vec{a})$$

The other way of writing it.

$$\delta(\vec{r}-\vec{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$$

In spherical coordinates, δ -function is given by

$$\delta(\vec{r}-\vec{r}') = \frac{1}{r^2} \delta(r-r') \delta(\cos\theta - \cos\theta') \delta(\phi-\phi')$$

$$= \frac{1}{r^2 \sin\theta} \delta(r-r') \delta(\theta-\theta') \delta(\phi-\phi')$$

Here we have used property of δ -function

$$\delta(\cos\theta - \cos\theta') = \frac{\delta(\theta-\theta')}{\sin\theta}$$

Fourier transform of 3d delta function.

$$\delta(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}$$

Two important relations encountered in electrodynamics involving δ -function -

$$\nabla \cdot \left(\frac{\vec{r}}{r^2} \right) = 4\pi \delta(\vec{r})$$

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi \delta(\vec{r})$$

Dimensions: Physical dimension = $\frac{1}{[\text{dimension of its argument}]}$

For 1-d, δ -function $[\delta(x)] = \frac{1}{[x]} = \frac{1}{[L]}$

2-d - $[\delta(x)\delta(y)] = \frac{1}{[L][L]} = \frac{1}{[L^2]}$

3-d - $[\delta(\vec{r})] = [\delta(x)\delta(y)\delta(z)] = \frac{1}{[L]} \cdot \frac{1}{[L]} \cdot \frac{1}{[L]} = \frac{1}{[L^3]}$

H.W. Evaluate the following integrals.

(I) $\int_0^5 \cos x \delta(x - \pi) dx$

(II) $\int_0^3 x^3 \delta(x+1) dx$

(III) $\int_{-\infty}^{+\infty} \ln(x+3) \delta(x+2) dx$

(IV) $\int_{-2}^2 (2x+3) \delta(3x) dx$

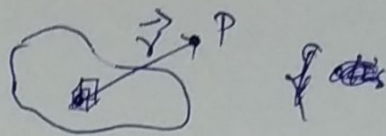
V $\int_0^2 (x^3 + 3x + 2) \delta(1-x) dx$

(VI) $\int_{-\infty}^a \delta(x-b) dx$

Continuous Charge distribution:

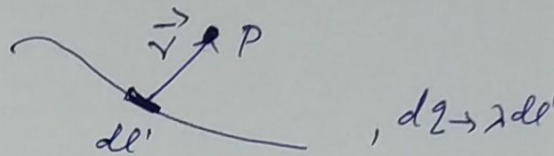
Continuous distribution -

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\vec{r}}{r^2} dq$$



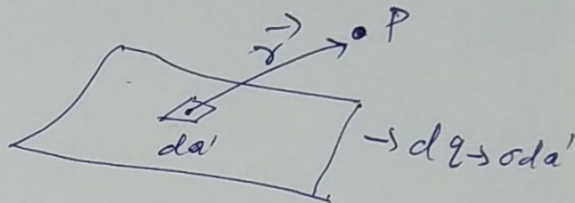
Line charge (λ) -

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}')}{r^2} \hat{r} dl'$$



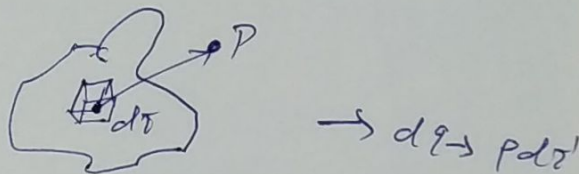
Surface charge (σ) -

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\vec{r}')}{r^2} \hat{r} da'$$



Volume charge (ρ) -

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{r^2} \hat{r} d\tau'$$



One can also write charge distribution say ($\tilde{\rho}(\vec{x})$) in terms of δ -function.

General form of \vec{E} in terms of $\tilde{\rho}(\vec{x})$ -

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \tilde{\rho}(\vec{x}') d\vec{x}'$$

For Point charge: $\tilde{\rho}(\vec{x}) = q_1 \delta^3(\vec{x} - \vec{x}_1)$, $x_0 \rightarrow$ Position of charge

Line charge: $\tilde{\rho}(\vec{x}) = \lambda(\vec{x}) \delta^2$

$\begin{cases} \vec{r} = \vec{x} - \vec{x}' \\ r = |\vec{x} - \vec{x}'| \end{cases}$
 $\delta^2 \rightarrow$ 2D delta function

Surface charge: $\tilde{\rho}(\vec{x}) = \sigma(\vec{x}) \delta^1$

$\delta^1(\vec{x}) = \delta(x - x_0) \delta(y - y_0)$ etc