B.Sc. (Hons.) Part III V^{th} Paper

METRIC SPACES (CONTD.)

EXAMPLES OF Metric Spaces:

1. If $x = (x_1, x_2)$, $y = (y_1, y_2)$ belong to R^2 then the function

$d : R^2 \times R^2 \to R$ defined as

$$d(x,y) = +\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$ is a metric on R^2. This is called

USUAL METRIC ON R^2.

Proof

1. obviously $(x_1-y_1)^2$ and $(x_2-y_2)^2$ are (+)ve.

$\Rightarrow \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ is always positive

$\Rightarrow d(x,y) \geq 0$

2. we prove that
$d(x,y) = 0 \Leftrightarrow x = y$.

Now, $d(x,y) = 0$

$\Leftrightarrow \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} = 0$

$\Leftrightarrow (x_1-y_1)^2 + (x_2-y_2)^2 = 0$

sum of two squares is zero iff

each of the square is equal to 0.

$\Leftrightarrow (x_1-y_1)^2=0,\ (x_2-y_2)^2=0$

$\Leftrightarrow x_1-y_1=0,\ x_2-y_2=0$

$\Leftrightarrow x_1=y_1,\ x_2=y_2.$

$\Leftrightarrow (x_1,x_2)=(y_1,y_2)$

$\Leftrightarrow x=y.$

Thus $d(x,y)=0 \Leftrightarrow x=y$

3. we shall prove that

$d(x,y)=d(y,x)$

$\because d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

$=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$

$=d(y,x)$

$\Rightarrow d(x,y)=d(y,x)$

4. we show that

$d(x,y)\leq d(x,z)+d(z,y)$

Given that $x=(x_1,x_2)$, $y=(y_1,y_2)$.

Let $z=(z_1,z_2)$.

Now $d(x,z)=\sqrt{(x_1-z_1)^2+(x_2-z_2)^2}$ —(1)

$d(z,y)=\sqrt{(z_1-y_1)^2+(z_2-y_2)^2}$ —(2)

Let $x_1 - z_1 = \alpha_1$, $x_2 - z_2 = \alpha_2$

$z_1 - y_1 = \beta_1$, $z_2 - y_2 = \beta_2$.

$\therefore$ (1) and (2) $\Rightarrow$ $d(x,z) = \sqrt{\alpha_1^2 + \alpha_2^2}$ ——(3)

and $d(z,y) = \sqrt{\beta_1^2 + \beta_2^2}$ ——(4)

Now, $x_1 - y_1 = (x_1 - z_1) + (z_1 - y_1) = \alpha_1 + \beta_1$ ——(5)

$x_2 - y_2 = (x_2 - z_2) + (z_2 - y_2) = \alpha_2 + \beta_2$ ——(6)

we've to prove ~~that~~

$d(x,y) \le d(x,z) + d(z,y)$

i.e. $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \le \sqrt{\alpha_1^2 + \alpha_2^2} + \sqrt{\beta_1^2 + \beta_2^2}$

[using (3) and (4)]

i.e. $\sqrt{(\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2} \le \sqrt{\alpha_1^2 + \alpha_2^2} + \sqrt{\beta_1^2 + \beta_2^2}$

[using (5) and (6)]

i.e. $(\alpha_1 + \beta_1)^2 + (\alpha_2 + \beta_2)^2 \le \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 + 2\sqrt{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)}$

i.e. $2(\alpha_1\beta_1 + 2\alpha_2\beta_2) \le 2\sqrt{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)}$

i.e. $[\alpha_1\beta_1 + \alpha_2\beta_2)^2 \le (\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)$

i.e. $\cancel{\alpha_1^2\beta_1^2} + \cancel{\alpha_2^2\beta_2^2} + 2\alpha_1\alpha_2\beta_1\beta_2$

$\le \cancel{\alpha_1^2\beta_1^2} + \cancel{\alpha_2^2\beta_2^2} + \alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2$

i.e. we've to prove that

$$2\alpha_1\alpha_2\beta_1\beta_2 \leq \alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2$$

~~But GM of~~

i.e. $$\frac{\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2}{2} \geq \alpha_1\alpha_2\beta_1\beta_2$$

Now AM of $\alpha_1^2\beta_2^2$ and $\alpha_2^2\beta_1^2$

$$= \frac{\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2}{2}$$

and their GM $= \sqrt{\alpha_1^2\beta_2^2 \cdot \alpha_2^2\beta_1^2}$

$$= \alpha_1\alpha_2\beta_1\beta_2$$

i.e. we've to show that

AM $\geq$ GM

which, we know that, is true.

Hence all the conditions for a metric space is satisfied.

Hence, d is a metric on R^2.