Summation of series (contd.)

1. Find the sum of the series

$\cos\alpha + 2\cos 2\alpha + 3\cos 3\alpha + \cdots + n\cos n\alpha.$

Soln.

Let $C = \cos\alpha + 2\cos 2\alpha + 3\cos 3\alpha + \cdots + n\cos n\alpha.$

Now, let $S = \sin\alpha + 2\sin 2\alpha + 3\sin 3\alpha + \cdots + n\sin n\alpha.$

$$\Rightarrow C+iS = (\cos\alpha + i\sin\alpha) + 2(\cos 2\alpha + i\sin 2\alpha) + 3(\cos 3\alpha + i\sin 3\alpha) + \cdots + n(\cos n\alpha + i\sin n\alpha)$$

$$\Rightarrow C+iS = e^{i\alpha} + 2e^{2i\alpha} + 3e^{3i\alpha} + \cdots + ne^{ni\alpha} \quad \text{---(1)}$$

Multiplying both sides by $e^{i\alpha}$, we get

$$(C+iS)e^{i\alpha} = e^{2i\alpha} + 2e^{3i\alpha} + 3e^{4i\alpha} + \cdots + (n-1)e^{ni\alpha} + ne^{(n+1)i\alpha} \quad \text{---(2)}$$

Subtracting (2) from (1), we get

$$\Rightarrow (C+iS)(1-e^{i\alpha}) = e^{i\alpha} + e^{2i\alpha} + e^{3i\alpha} + \cdots + e^{ni\alpha} - ne^{(n+1)i\alpha}$$

$$\Rightarrow (C+iS)(1-e^{i\alpha}) = \frac{e^{i\alpha}(1-e^{ni\alpha})}{1-e^{i\alpha}} - ne^{(n+1)i\alpha}$$

$$\Rightarrow (C+iS)(1-e^{i\alpha}) = \frac{1-e^{ni\alpha}}{e^{-i\alpha}(1-e^{i\alpha})} - ne^{(n+1)i\alpha}$$

$$\Rightarrow (C+iS)(1-e^{i\alpha}) = \frac{-(1-e^{ni\alpha})}{1-e^{-i\alpha}} - ne^{(n+1)i\alpha}$$

$$\Rightarrow C+iS = -\frac{1-e^{ni\alpha}}{(1-e^{-i\alpha})(1-e^{i\alpha})} - \frac{n\cdot e^{i(n+1)\alpha}}{1-e^{i\alpha}}$$

$$\Rightarrow C+iS = -\frac{1-e^{ni\alpha}}{1-(e^{i\alpha}+e^{-i\alpha})+1} - \frac{n\,e^{i(n+1)\alpha}}{(1-e^{i\alpha})} \times \left(\frac{1-e^{-i\alpha}}{1-e^{-i\alpha}}\right)$$

$$\Rightarrow C+iS = \frac{e^{ni\alpha}-1}{2-2\cos\alpha} - \frac{n\left[e^{i(n+1)\alpha} - e^{ni\alpha}\right]}{2-2\cos\alpha}$$

$$\Rightarrow C+iS = \frac{(\cos n\alpha + i\sin n\alpha - 1) - n\left[\cos(n+1)\alpha + i\sin(n+1)\alpha - \cos n\alpha - i\sin n\alpha\right]}{2-2\cos\alpha}$$

Equating real parts from both sides, we get

$$\Rightarrow C = \frac{\cos n\alpha - 1 - n\cos(n+1)\alpha + n\cos n\alpha}{2(1-\cos\alpha)}$$

$$\Rightarrow C = \frac{(n+1)\cos n\alpha - n\cos(n+1)\alpha - 1}{2(1-\cos\alpha)}$$

This is the required sum of the given series.

2. Find the sum of the series

$$1+\frac{2}{2}\cos\theta+\frac{3}{2^2}\cos 2\theta+\frac{4}{2^3}\cos 3\theta+\cdots \text{ ad.inf.}$$

Soln

Let us denote the given series by C.

$$\Rightarrow C = 1+\frac{2}{2}\cos\theta+\frac{3}{2^2}\cos 2\theta+\frac{4}{2^3}\cos 3\theta+\cdots \text{ to } \infty$$

Let $S = \frac{2}{2}\sin\theta+\frac{3}{2^2}\sin 2\theta+\frac{4}{2^3}\sin 3\theta+\cdots \text{ to } \infty$

$$\Rightarrow C+iS = 1+\frac{2}{2}(\cos\theta+i\sin\theta)+\frac{3}{2^2}(\cos 2\theta+i\sin 2\theta)$$
$$+\frac{4}{2^3}(\cos 3\theta+i\sin 3\theta)+\cdots \text{ to } \infty$$

$$\Rightarrow C+iS = 1+\frac{2}{2}e^{i\theta}+\frac{3}{2^2}e^{2i\theta}+\frac{4}{2^3}e^{3i\theta}+\cdots \text{ to } \infty \quad \text{--- (1)}$$

multiplying both sides of (1) by $\frac{1}{2}e^{i\theta}$, we get

$$\Rightarrow (C+iS)\frac{e^{i\theta}}{2} = \frac{e^{i\theta}}{2}+\frac{2}{2^2}e^{2i\theta}+\frac{3}{2^3}e^{3i\theta}+\frac{4}{2^4}e^{4i\theta}+\cdots \text{ to } \infty \quad \text{--- (2)}$$

subtracting (2) from (1), we get

$$\Rightarrow (C+iS)\left(1-\frac{1}{2}e^{i\theta}\right) = 1+\frac{1}{2}e^{i\theta}+\frac{1}{2^2}e^{2i\theta}+\frac{1}{2^3}e^{3i\theta}+\cdots \text{ to } \infty$$

$$\Rightarrow (C+iS)\left(\frac{2-e^{i\theta}}{2}\right) = \frac{1}{1-\frac{1}{2}e^{i\theta}}$$ [The series of RHS is in G.P.]

$$\Rightarrow (C+iS)\frac{(2-e^{i\theta})}{2} = \frac{2}{2-e^{i\theta}}$$

$$\Rightarrow C+iS = \frac{4}{(2-e^{i\theta})^2}$$

$$\Rightarrow C + iS = \frac{4}{(2-e^{i\theta})^2} \times \frac{(2-e^{-i\theta})^2}{(2-e^{-i\theta})^2}$$

$$\Rightarrow C + iS = \frac{4\left[2-e^{-i\theta}\right]^2}{\left[(2-e^{i\theta})(2-e^{-i\theta})\right]^2}$$

$$\Rightarrow C + iS = \frac{4\left[4 + e^{-2i\theta} - 4e^{-i\theta}\right]}{\left[4 - 2(e^{i\theta}+e^{-i\theta}) + 1\right]^2}$$

$$\Rightarrow C + iS = \frac{4\left[4 + \cos 2\theta - i\sin 2\theta - 4\cos\theta + 4i\sin\theta\right]}{(5-4\cos\theta)^2}$$

Equating real parts, we get

$$\Rightarrow C = \frac{4(4+\cos 2\theta - 4\cos\theta)}{(5-4\cos\theta)^2}$$