Divergence Theorem due to Gauss

(Mathematical Physics, B.Sc. (Physics) Part-III, Paper-V)

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(Dated: July 16, 2020)

"Mathematics is, in its way, the poetry of logical ideas."

— Albert Einstein (1879-1955)

In the earlier lecture notes, we have discussed vector integration (line integral, surface integral and volume integral). Here we discuss divergence theorem which relates surface integral to volume integral. In mathematical form it is written, for a vector field $\mathbf{V}$, as

$$\oiint_s \mathbf{V} \cdot \mathbf{da} = \iiint_v \nabla \cdot \mathbf{V} \, d\tau \ , \tag{1}$$

which states that surface integral of the vector field $\mathbf{V}$ over the boundary of a closed surface (valid for any arbitrary shape of surface) is equal to the volume integral of the divergence of $\mathbf{V}$ over the enclosed volume

Proof: Let's take right-hand side integral and expand it

$$\iiint_v \nabla \cdot \mathbf{V} \, d\tau = \iiint_v \Big(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \Big) \, dx \, dy \, dz \ . \tag{2}$$

Let us consider a cubical shape surface to prove the theorem (Fig1). Note that for sign convention we take outward normal to a closed surface to be positive.

In the Eq. (1), $\mathbf{da} = \hat{\mathbf{n}} da$, where $\hat{\mathbf{n}}$ is the unit outward normal to the area element da.

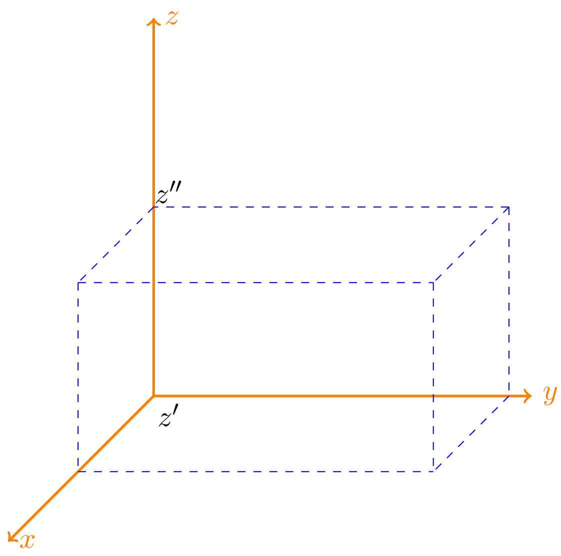

FIG. 1: Cube.

The area element for all the six sides of the cube are given by

$$\text{(bottom)}\quad \mathbf{da}_1 = -\mathbf{k}\,\mathrm{dx}\,\mathrm{dy},$$
$$\text{(front)}\quad \mathbf{da}_2 = \mathbf{i}\,\mathrm{dy}\,\mathrm{dz},$$
$$\text{(left)}\quad \mathbf{da}_3 = -\mathbf{j}\,\mathrm{dx}\,\mathrm{dz},$$
$$\text{(back)}\quad \mathbf{da}_4 = -\mathbf{i}\,\mathrm{dy}\,\mathrm{dz},$$
$$\text{(top)}\quad \mathbf{da}_5 = \mathbf{k}\,\mathrm{dx}\,\mathrm{dy},$$
$$\text{(right)}\quad \mathbf{da}_6 = \mathbf{j}\,\mathrm{dx}\,\mathrm{dz}$$

Now let us integrate the last term on the rhs of Eq. (1) w.r.t. z from z' to z''.

$$\begin{aligned}\iiint_{z'}^{z''} \frac{\partial V_z}{\partial z}\,dx\,dy\,dz &= \iint_a (V_z(x,y,z'') - V_z(x,y,z'))\,dx\,dy \\ &= \iint_{a_5} V_z(x,y,z'')\,dx\,dy - \iint_{a_1} V_z(x,y,z')\,dx\,dy.\end{aligned}$$

Here for the bottom and top sides we have respectively $\hat{\mathbf{n}}_1 \cdot \mathbf{k}\,da_1 = -dx\,dy$ and $\hat{\mathbf{n}}_5 \cdot \mathbf{k}\,da_5 = dx\,dy$. Now using these expressions in the above Eq. (3) we obtain

$$\begin{aligned}\iiint_{z'}^{z''} \frac{\partial V_z}{\partial z}\,dx\,dy\,dz &= \iint_{a_5} V_z(x,y,z'')\hat{\mathbf{n}}_5 \cdot \mathbf{k}\,da_5 + \iint_{a_1} V_z(x,y,z')\hat{\mathbf{n}}_1 \cdot \mathbf{k}\,da_1 \\ &= \oiint_a V_z \mathbf{k} \cdot \hat{\mathbf{n}} da\ .\end{aligned} \tag{3}$$

The contribution from the other sides are taken to be zero since $\mathbf{k}$ is perpendicular to surface elements $d\mathbf{a}_2$, $d\mathbf{a}_3$, $d\mathbf{a}_4$ and $d\mathbf{a}_6$. Next, using the similar argument we can write

$$\iiint_v \frac{\partial V_z}{\partial z}\, d\tau = \oiint_a V_x \mathbf{i} \cdot \hat{\mathbf{n}} da \ , \tag{4}$$

and

$$\iiint_v \frac{\partial V_y}{\partial z}\, d\tau = \oiint_a V_z \mathbf{k} \cdot \hat{\mathbf{n}} da \ . \tag{5}$$

Now using Eqs. (3), (4) and (5), we can write

$$\boxed{\oiint_s \mathbf{V} \cdot \mathbf{da} = \iiint_v \nabla \cdot \mathbf{V}\, d\tau} \ . \tag{6}$$

[1] G. B. Arfken, and H. J. Weber, *Mathematical methods for physicist*, Elsevier Academic Press, (2005) .

[2] M. L. Boas, *Mathematical Methods in Physical Sciences*, Wiley, (1966).

[3] C. Harper, *Introduction to mathematical physics*, Prentice-Hall of India, (2003).