Divergence of Electric Field

(B.Sc. (Physics) Part-II, Paper-IV, Group-A)

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"No one undertakes research in physics with the intention of winning a prize. It is the joy of discovering something no one knew before."
— Stephen Hawking (1942-2018)

In the last lecture note, we have obtained Gauss's law in integral and differential form. The differential form of Gauss's law states that divergence of electric field is equal to charge density divided by permitivity of free space. Here we obtain the divergence of **E** directly from Coulomb's law

Coulomb's law for charge density ρ is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_{all\ \ space} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')\, d\mathbf{x}' \ . \tag{1}$$

Note that here $d\mathbf{x}' \equiv dx'\, dy'\, dz'$. Next, we take divergence of above equation

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{all\ \ space} \nabla\cdot\left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\rho(\mathbf{x}')\, d\mathbf{x}' \ . \tag{2}$$

It is straightforward to see that

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|} \ . \tag{3}$$

Now using above expression in Eq. (2), we can write

$$\nabla\cdot\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\int_{all\ \ space} \nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\rho(\mathbf{x}')\, d\mathbf{x}' \ . \tag{4}$$

Next, we use the relation

$$\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}').$$

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To prove the above expression see B.Sc. Part-III, Mathematical Physics notes uploaded on the website. If you have any confusion kindly email me. My email address is given at the bottom of page one of this class note.

Now we can write Eq. (4)

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_{all\ \ space} 4\pi\delta(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')\, d\mathbf{x}' \\ &= \frac{1}{\epsilon_0} \int_{all\ \ space} \delta(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')\, d\mathbf{x}' \ . \end{aligned} \tag{5}$$

Now using the property of delta function in above expression, we obtain

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{x})}{\epsilon_0}} \ , \tag{6}$$

which is the Gauss's law obtained in the last lecture note.

[1] D.J. Griffiths, *Introduction to Electrodynamics*, New Jersey: Prentice Hall, (1999) .

[2] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, (1962).