The Curl of Electric field

(B.Sc. (Physics) Part-II, Paper-IV, Group-A)

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"Look up at the stars and not down at your feet. Try to make sense of what you see, and wonder about what makes the universe exist. Be curious."

— Stephen Hawking (1942-2018)

In the last lecture note, we have obtained Gauss's law in differential form by taking the divergence of electric field **E** directly from Coulomb's law. Here we obtain the curl of **E** .

But before moving to that topic, here I want to point out that Gauss's law in integral form gives us powerful tool to calculate electric field easily provided that one can identify the symmetry in the given problem. For example, one wants to calculate the electric field due to a uniformly charged body of spherical symmetry, then he should choose the Gaussian surface accordingly. Mainly we will have three type of symmetry which will be applicable to evaluate **E** using Gauss's law. These are spherical, cylindrical and plane symmetry. We will elaborate this more when we discuss various problems on Gauss's law and their simple solutions.

Now we move to the main topic of today's discussion, i.e., to obtain curl of **E** . Coulomb's law for a point charge q is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} q \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{1}$$

Note that we can define $|\mathbf{x}-\mathbf{x}'| = r$ for simplicity of notation. If we put the charge at the origin, we obtain

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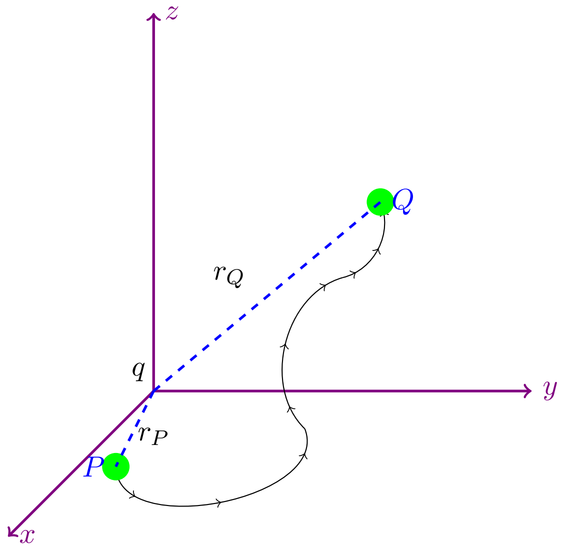

FIG. 1: (a)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \; . \tag{2}$$

Next, we evaluate $\int_P^Q \mathbf{E}\cdot d\mathbf{l}$ (see Fig. (1)). We choose spherical coordinate and write length element in the form

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \; . \tag{3}$$

Now we dot $\mathbf{E}$ with $d\mathbf{l}$ and obtain

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr \; . \tag{4}$$

Now

$$\begin{aligned}\int_P^Q \mathbf{E}\cdot d\mathbf{l} &= \frac{1}{4\pi\epsilon_0}\int_P^Q \frac{q}{r^2}\,dr \\ &= -\frac{1}{4\pi\epsilon_0}\left[\frac{q}{r}\right]_P^Q \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_P}-\frac{1}{r_Q}\right] \; .\end{aligned} \tag{5}$$

For closed path (see Fig. (1)), the distance $r_P = r_Q$, where r_P is the distance of point P from origin and r_Q is the distance of point Q from origin. Therefore, we can write

$$\oint_P^Q \mathbf{E}\cdot d\mathbf{l} = 0 \; . \tag{6}$$

Applying Stokes' theorem (*click the blue link for details of Stokes' theorem*)

http://maharajacollege.ac.in/fileupload/uploads/5f191621e164420200723044625Stokes%20theorem.pdf

$$\iint_{\mathcal{A}} (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}}\, da = \oint_{\mathcal{L}} \mathbf{E} \cdot d\mathbf{l} \;, \tag{7}$$

we can write

$$\boxed{\nabla \times \mathbf{E} = 0 \;.} \tag{8}$$

The above result also hold for arbitrary choice of reference frame and location of charge as well as Eq. (8) also applicable of many charges. Since in case of many charges the electric field will be obtained using the principle of superposition. The curl of $\mathbf{E}$ will be zero. *Note that Eqs. (6) and (8) holds for static charge distribution.*

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[2] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, (1962).