The Cauchy Condensation Test
(named for Augustin Louis Cauchy, 1789–1857)

The Cauchy Condensation Test (CCT) is as follows:

Suppose that $a_n > 0$ *for all* $n \geq 1$ *and that* a_n *is monotone decreasing. Then the series*

$$\sum a_n \quad \text{and} \quad \sum 2^n a_{2^n}$$

either both converge or both diverge.

Before proving this theorem, let us illustrate it with some examples.

Example 1: Let us use the CCT to show that the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, diverges.

For this series, we have $a_n = 1/n$ and it is obvious that $a_n > 0$ for all $n \geq 1$ and that a_n is monotone decreasing. Now notice that

$$a_{2^n} = \frac{1}{2^n}$$

and thus

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n}\right) = \sum_{n=0}^{\infty} 1.$$

Since this series clearly diverges (because it is a sum of 1s), then the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges by the CCT.

Example 2: Let us use the CCT to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

For this series, we have $a_n = 1/n^2$ and it is obvious that $a_n > 0$ for all $n \geq 1$ and that a_n is monotone decreasing. Now notice that

$$a_{2^n} = \frac{1}{(2^n)^2} = \frac{1}{2^{2n}} = \frac{1}{4^n}$$

and thus

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{4^n}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.$$

Since this series converges (because it is a geometric series with $x = 1/2$), then the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges by the CCT.

Example 3: Let us use the CCT to show that the series $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges.

For this series, we have $a_n = \frac{1}{n\ln(n)}$ and it is obvious that $a_n > 0$ for all $n \geq 2$ and that a_n is monotone decreasing. Now notice that

$$a_{2^n} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n n \ln(2)}$$

and thus

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^n n \ln(2)}\right) = \sum_{n=1}^{\infty} \frac{1}{\ln(2)} \cdot \frac{1}{n}.$$

Since this series clearly diverges (because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the above series is just this series with every term multiplied by $1/\ln(2)$), then the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ also diverges by the CCT.

We will now give the proof of the CCT:

Let S_n be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and let T_n be the sequence of partial sums of $\sum_{n=1}^{\infty} 2^n a_{2^n}$.Thus

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

for all $n \geq 1$ and

$$T_n = 2a_2 + 4a_4 + 8a_8 + \cdots + 2^n a_{2^n}$$

for all $n \geq 1$.

Because all of the numbers a_n are positive and the sequence a_n is monotone decreasing, notice that

$$\begin{aligned}
2a_2 + T_1 &= 2a_2 + 2a_2 \\
&\leq 2a_1 + 2a_2 \\
&= 2(a_1 + a_2) \\
&= 2S_2
\end{aligned}$$

This shows that $2a_2 + T_1 \leq 2S_2$.

Now notice that

$$\begin{aligned}
2a_2 + T_2 &= 2a_2 + 2a_2 + 4a_4 \\
&= 2a_2 + 2a_2 + 2a_4 + 2a_4 \\
&\leq 2a_1 + 2a_2 + 2a_3 + 2a_4 \\
&= 2(a_1 + a_2 + a_3 + a_4)
\end{aligned}$$

This shows that $2a_2 + T_2 \leq 2S_4$.

Continuing with this type of reasoning, we can see that

$$2a_2 + T_n \leq 2S_{2^n}$$

for all $n \geq 1$ or, equivalently, that

$$T_n \leq 2S_{2^n} - 2a_2$$

Now suppose That $\sum a_n$ converges. Then $\lim_{n\to\infty} S_n$ exists (say that $\lim_{n\to\infty} S_n = L$). This means that is is also true that $\lim_{n\to\infty} S_{2^n} = L$ and hence that

$$\lim_{n\to\infty} (2S_{2^n} - 2a_2) = 2L - 2a_2.$$

Therefore $T_n \leq 2L - 2a_2$ for all $n \geq 1$. Since T_n is monotone increasing (because $2^n a_{2^n} > 0$ for all $n \geq 1$) and T_n is bounded above (by $2L - 2a_2$), then $\lim_{n\to\infty} T_n$ exists and hence $\sum 2^n a_{2^n}$ converges. We have now proved that if $\sum a_n$ converges, then $\sum 2^n a_{2^n}$ also converges.

To complete the proof, we must show that if $\sum a_n$ diverges, then $\sum 2^n a_{2^n}$ also diverges. To do this, note that

$$\begin{aligned} a_1 + T_1 &= a_1 + 2a_2 \\ &= a_1 + a_2 + a_2 \\ &\geq a_1 + a_2 + a_3 \\ &= S_3. \end{aligned}$$

This shows that $a_1 + T_1 \geq S_3$.

Next note that

$$\begin{aligned} a_1 + T_2 &= a_1 + 2a_2 + 4a_4 \\ &= a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 \\ &\geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \\ &= S_7. \end{aligned}$$

This shows that $a_1 + T_2 \geq S_7$.

Continuing with this type of reasoning, we can in fact verify that

$$a_1 + T_n \geq S_{2^{n+1}-1}$$

for all $n \geq 1$ or equivalently, that

$$S_{2^{n+1}-1} - a_1 \leq T_n$$

for all $n \geq 1$.

Now suppose That $\sum a_n$ diverges. Then $\lim_{n\to\infty} S_n = \infty$ (because $a_n > 0$ for all n). It follows that also $\lim_{n\to\infty} S_{2^{n+1}-1} = \infty$ and hence that $\lim_{n\to\infty} (S_{2^{n+1}-1} - a_1) = \infty$. By the inequality given above, we conclude that $\lim_{n\to\infty} T_n = \infty$ and hence that $\sum 2^n a_{2^n}$ diverges.

This completes the proof of the Cauchy Condensation Test.

Let us look at just one more example:

Example 4: Does the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)}$ converge or diverge?

We will use the CCT to answer this question. To do this, we first observe that

$$a_n = \frac{1}{n\ln(n^3)} > 0$$

for all $n \geq 2$ and that a_n is monotone decreasing (because $n\ln(n^3)$ is monotone increasing).

Now we observe that

$$a_{2^n} = \frac{1}{2^n \ln\left((2^n)^3\right)} = \frac{1}{2^n \ln(8^n)} = \frac{1}{2^n n \ln(8)}$$

and hence that

$$\sum 2^n a_{2^n} = \sum 2^n \left(\frac{1}{2^n n \ln(8)}\right) = \sum \frac{1}{\ln(8)} \cdot \frac{1}{n}.$$

Since $\sum \frac{1}{n}$ diverges, then $\sum \frac{1}{\ln(8)} \cdot \frac{1}{n}$ also diverges and we conclude that $\sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)}$ diverges by the CCT.