Angular Momentum Operator

A Particle of mass m revolving around a point P at a distance r.
The angular momentum of Particle is L.

$$L = mur = mr^2\omega = I\omega \quad \text{——— (1)}$$

where u is linear velocity, ω is angular velocity $(\omega = \frac{u}{r})$ and moment of inertia $I = mr^2$.

The Particle has Kinetic energy E_K.

$$\therefore E_K = \frac{1}{2}mu^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$$

$$E_K = \frac{1}{2}\frac{I\cdot\omega \times I}{I} = \frac{1}{2}\frac{(I\omega)^2}{I} = \frac{L^2}{2I}$$

In terms of vector, $\vec{L} = \vec{r}\cdot m\vec{u} = \vec{r}\cdot\vec{p}$

where $\vec{r}$ is radial distance vector and $\vec{p}$ is linear momentum vector.
The vector $\vec{L}$ is perpendicular to the plane defined by $\vec{r}$ and $\vec{p}$.

From Classical mechanics,
The component of the classical angular momentum vector are given by

$$\left.\begin{aligned} L_x &= yP_z - zP_y \\ L_y &= zP_x - xP_z \\ L_z &= xP_y - yP_x \end{aligned}\right\} \quad \text{——— (1)}$$

The square of angular momentum is given by the scalar product of $\vec{L}$ with itself.

Thus, $L^2 = \vec{L}\cdot\vec{L} = L_x^2 + L_y^2 + L_z^2$

In classical mechanics it permits all possible values of L.

But in quantum mechanics the angular momentum is represented by an Operator.

The quantum mechanical Operators for the components of angular momentum are obtained by replacing the quantities in equation (1) with their corresponding quantum mechanical Operators.

$$\text{Thus,} \quad \hat{P}_x = i\hbar\frac{\partial}{\partial x} \quad \text{and ... so on - - - -}$$

$$\text{So,} \quad \hat{L}_x = i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad \text{——— (2)}$$

$$\hat{L}_y = i\hbar\left(z\frac{\partial}{\partial x} - x\cdot\frac{\partial}{\partial z}\right) \quad \text{——— (3)}$$

$$\text{and} \quad \hat{L}_z = i\hbar\left(x\frac{\partial}{\partial y} - y\cdot\frac{\partial}{\partial x}\right) \quad \text{——— (4)}$$

The Operator for the square of angular momentum is given by

$$\hat{L}^2 = |\vec{L}|^2 = \vec{L}\cdot\vec{L} = L_x^2 + L_y^2 + L_z^2 \quad \text{——— (5)}$$

It is important that the only square of angular momentum L^2 and one of its component only can be measured simultaneously.
i.e L^2 and L_x or L^2 and L_y or L^2 and L_z can measured simultaneously. This fact is expressed by saying that L^2 Commutes with one of its Components.

$$[\hat{L}^2 . \hat{L}_x] = [\hat{L}^2 . \hat{L}_y] = [\hat{L}^2 . \hat{L}_z] = 0 \quad \text{——— (6)}$$

On the other hand $[\hat{L}_x, \hat{L}_y]$, $[\hat{L}_y, \hat{L}_z]$, $[\hat{L}_z, \hat{L}_x]$ can not measured simultaneously. In terms of Quantum mechanics these Operator does not commute with each other

$$\text{or} \quad [\hat{L}_x, \hat{L}_y] \neq 0, \; [\hat{L}_y, \hat{L}_z] \neq 0 \; \text{or} \; [\hat{L}_z, \hat{L}_x] \neq 0 \quad \text{—}$$

Linear Operator

An Operator $\hat{A}$ is said to be linear if its application on the sum of two functions gives the result which is equal to the sum of the operations on the two functions separately.

i.e. $$\hat{A}\left[f(x)+g(x)\right] = \hat{A} f(x) + \hat{A} g(x)$$

or $$\hat{A} \cdot C f(x) = C \hat{A} f(x) \quad , C = \text{Constant}$$

The Operator $\frac{d}{dx}$ is linear Operator

because, $$\frac{d}{dx}\left(ax^m + bx^n\right) = \frac{d}{dx}\left(ax^m\right) + \frac{d}{dx}\left(bx^n\right)$$

On the Other hand, Square root (symbol $\sqrt{\ }$) is not a linear Operator.

because $$\sqrt{g(x)+f(x)} \neq \sqrt{g(x)} + \sqrt{f(x)}$$

Or, $$\sqrt{f(x)+g(x)} \neq \sqrt{f(x)} + \sqrt{g(x)}$$

Hermitian Operator: -

· An Operator $\hat{A}$ having two eigenfunctions ψ_1 and ψ_2 are said to be Hermitian.

· When, $\int \psi_1 (\hat{A} \psi_2) d\tau = \int \psi_2 (\hat{A} \psi_1) d\tau$

where ψ_1 and ψ_2 are real.

or, $\int \psi_1^* (\hat{A} \psi_2) d\tau = \int \psi_2 (\hat{A} \psi_1)^* d\tau$

When ψ_1 and ψ_2 are Complex. Here ψ_1^* is the complex Conjugate of ψ_1 and $d\tau$ is the volume element of space in which the functions ψ_1 and ψ_2 are defined.

The functions e^{ix} (ψ_1) and $\sin x$ (ψ_2) are the two acceptable eigen functions of Operator $\frac{d^2}{dx^2}$ ($\hat{A}$)

Now $\int \psi_1^* (\hat{A} \psi_2) d\tau = \int e^{-ix} \left(\frac{d^2}{dx^2} \sin x\right) dx$

$$= - \int e^{ix} \sin x \, dx \qquad \text{———} (1)$$

and $\int \psi_2 (A\psi_1)^* d\tau = \int \sin x \left[\frac{d^2 (e^{ix})^*}{dx^2}\right] dx$

$$= \int \sin x \, (i^2 e^{ix})^* dx$$

$$= - \int \sin x \, e^{-ix} dx \qquad \text{———} (2)$$

From equation (1) and (2)

$$\int \psi_1^* (\hat{A} \psi_2) d\tau = \int \psi_2 (A \psi_1)^* d\tau$$

Hence the Operator $\hat{A} = \frac{d^2}{dx^2}$ is a Hermitian Operator.

Show that the eigenvalues of a Hermitian Operator is real.

Let $\hat{A}$ is a Hermitian Operator with eigenfunction ψ and Corresponding eigen-value a.

Now we have,

$$\hat{A}\psi = a\psi \quad \text{——— (1)}$$

and

$$(\hat{A}\psi)^* = a^*\psi^* \quad \text{——— (2)}$$

On Multiplying the equation (1) with ψ^* and integrating we have,

$$\int \psi^* \hat{A} \psi \, d\tau = \int \psi^* a \psi \, d\tau \quad \text{(where } d\tau \text{ is volume element)}$$

$$= a \int \psi^* \psi \, d\tau \quad \text{——— (3)}$$

Again multiplying the equation (2) with ψ and then integrating we have,

$$\int \psi (\hat{A}\psi)^* d\tau = \int \psi a^* \psi^* d\tau$$

$$= a^* \int \psi^* \psi \, d\tau \quad \text{——— (4)}$$

Since $\hat{A}$ is a Hermitian Operator,

$$\therefore \int \psi^* \hat{A} \psi \, d\tau = \int \psi (\hat{A}\psi)^* d\tau \quad \text{——— (5)}$$

Now from equations (3), (4), and (5) we get

$$a \int \psi \psi^* d\tau = a^* \int \psi \psi^* d\tau$$

$$\text{or,} \quad a = a^*$$

It is possible only when a is real,

Thus we can say a Hermitian Operator has real eigenvalues.

The eigenfunctions of a Hermitian Operator Corresponding to different eigen Values are Orthogonal.

Let ψ_1 and ψ_2 are the two eigenfunctions of a Hermitian Operator $\hat{A}$ With Corresponding eigen values a_1 and a_2 respectively.

The eigenvalue equations are

$$\hat{A}\psi_1 = a_1\psi_1 \quad \text{—— (1)}$$

and

$$\hat{A}\psi_2 = a_2\psi_2 \quad \text{—— (2)}$$

On multiplying equation (1) with ψ_2^* and then integrating, we have:

$$\int \psi_2^* \hat{A} \psi_1 \, d\tau = \int \psi_2^* a_1 \psi_1 \, d\tau$$

$$= a_1 \int \psi_2^* \psi_1 \, d\tau \quad \text{—— (3)}$$

But Since $\hat{A}$ is a Hermitian Operator, we have

$$\int \psi_2^* \hat{A} \psi_1 d\tau = \int \psi_1 (\hat{A}\psi_2)^* d\tau$$

$$= \int \psi_1 a_2 \psi_2^* d\tau$$

$$= a_2 \int \psi_1 \psi_2^* d\tau \quad \text{—— (4)}$$

from equation (3) and (4)

$$a_1 \int \psi_2^* \psi_1 \, d\tau = a_2 \int \psi_1 \psi_2^* d\tau$$

$$\text{Or, } (a_1 - a_2) \int \psi_1 \psi_2^* d\tau = 0 \quad \text{—— (5)}$$

Since $a_1 \neq a_2$,

so, $a_1 - a_2 \neq 0$

$$\therefore \int \psi_1 \psi_2^* d\tau = 0$$

Hence ψ_1 and ψ_2 are Orthogonal.