M.Sc. IV th Semester

Mathematical methods

Legendre's differential equations

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

One of the solution of the Legendre's d.e. is known as $P_n(x)$.

And
$$P_n(x) = \frac{1.3.5\ldots(2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \cdots\right] \quad \text{—(1)}$$

This is called Legendre's polynomial of degree n.

$$\Rightarrow P_n(x) = \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{\lfloor 2n-2r}{2^n \lfloor r \lfloor n-r \lfloor n-2r} x^{n-2r}$$

where
$$\left[\frac{n}{2}\right] = \begin{cases} n/2, & n \text{ is even} \\ \frac{n-1}{2}, & n \text{ is odd} \end{cases}$$

From (1), we have

$$P_n(x) = \frac{1.2.3.4.5.6\ldots(2n-1)(2n)}{2.4.6\ldots2n.\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

$$\Rightarrow P_n(x) = \frac{\lfloor 2n}{(\lfloor n)^2 2^n} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{n-4} \cdots \right] \quad \text{---} (2)$$

Putting $n = 0$ in the eq (2), we have

$$P_0(x) = \frac{1}{1} \cdot [1 - 0] = 1.$$

i.e. $\boxed{P_0(x) = 1}$

Putting $n = 1$ in the eq (2), we get

$$P_1(x) = \frac{\lfloor 2}{1^2 \cdot 2} [x - 0] = x$$

i.e. $\boxed{P_1(x) = x}$

Putting $n = 2$ in the eq (2), we get

$$P_2(x) = \frac{\lfloor 4}{(\lfloor 2)^2 \cdot 2^2} \left[x^2 - \frac{2 \cdot 1}{2 \cdot 3} x^0 + 0 \right]$$

$$= \frac{24 \cdot 3}{16_2} \left[x^2 - \frac{1}{3} \right] = \frac{3x^2 - 1}{\cancel{3}} \times \frac{\cancel{3}}{2}$$

i.e. $\boxed{P_2(x) = \frac{3x^2 - 1}{2}}$