Q. $$\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x$$

Soln. $\rightarrow$ Given equation

$$(D^3 - 2D + 4)y = e^x \cos x$$

Auxiliary eqn is $D^3 - 2D + 4 = 0$

$$\Rightarrow D^3 + 2D^2 - 2D^2 - 4D + 2D + 4 = 0$$

$$\Rightarrow D^2(D+2) - 2D(D+2) + 2(D+2) = 0$$

$$\Rightarrow (D+2)(D^2 - 2D + 2) = 0$$

Either $D+2=0 \Rightarrow D=-2$

or $D^2 - 2D + 2 = 0 \Rightarrow (D-1)^2 + 1 = 0$

$$\Rightarrow (D-1)^2 = -1 = i^2$$

$$\therefore D-1 = \pm i$$

$$\Rightarrow D = 1+i,\ 1-i$$

$$\therefore CF = c_1 e^{-2x} + c_2 e^{(1+i)x} + c_3 e^{(1-i)x}$$

$$\Rightarrow CF = c_1 e^{-2x} + c_2 e^x \cdot e^{ix} + c_3 e^x \cdot e^{-ix}$$

$$= c_1 e^{-2x} + c_1 e^x(\cos x + i\sin x) + c_3 e^x(\cos x - i\sin x)$$

$$= c_1 e^{-2x} + e^x[(c_1 + c_3)\cos x + i(c_1 - c_3)\sin x$$

$$CF = c_1 e^{-2x} + e^x(A\cos x + B\sin x)$$

Now, we shall find P.I.

$$P.I. = \frac{1}{D^3 - 2D + 4} e^x \cos x$$

[We shall always first integrate e^x in such cases.
~~For~~ ~~this~~ we shall put $D = D + a$
where a is e^{ax}.]

$$= e^x \cdot \frac{1}{(D+1)^3 - 2(D+1) + 4} \cos x$$

$$= e^x \cdot \frac{1}{D^3 + 3D^2 + 3D + 1 - 2D - 2 + 4} \cos x$$

$$= e^x \cdot \frac{1}{D^3 + 3D^2 + D + 3} \cos x \quad X$$

$$= e^x \frac{1}{D \cdot D^2 + 3 \cdot D^2 + D + 3} \cos x$$

$$\therefore PI = e^{x} \cdot \frac{1}{D\cdot(-1)+3\times(-1)+D+3} \cos x$$

Here, the denominator becomes zero.

Impt: So, we shall not replace D^2 by $-a^2$

[as in the case of $\cos ax/\sin ax$]

Again, from (*)

$$PI = e^{x} \cdot \frac{1}{D^3+3D^2+D+3} \cos x$$

$$= e^{x} \cdot x \frac{1}{3D^2+6D+1} \cos x$$

[by differentiating the denominator and multiplying numerator by x

$$\Rightarrow PI = xe^{x} \cdot \frac{1}{3\times(-1)+6D+1} \cos x$$

$$= xe^{x} \cdot \frac{1}{6D-2} \cos x$$

$$= xe^{x} \frac{1}{2(3D-1)} \cos x$$

$\therefore PI = \frac{1}{2} x e^{x} \frac{3D+1}{(3D+1)(3D-1)} \cos x$

$\Rightarrow PI = \frac{1}{2} x e^{x} . \frac{(3D+1)}{9D^2 - 1} \cos x$

$= \frac{1}{2} x e^{x} \frac{3D(\cos x) + \cos x}{9 \times (-1) - 1}$

$\Rightarrow PI = -\frac{1}{20} x e^{x} (\cos x - 3 \sin x)$

Hence the complete soln is

$y = CF + PI$

$\Rightarrow y = C_1 e^{-2x} + e^{x} (A \cos x + B \sin x)$

$- \frac{1}{20} x e^{x} (\cos x - 3 \sin x)$.