Orthogonal Curvilinear coordinates:—

In previous lecture note we have discussed the metric coefficient g_{ij} is given by

$$g_{ij} = \vec{a_i}\cdot\vec{a_j}, \quad \text{—— (1)}$$

which is symmetric

For orthogonal curvilinear coordinates $g_{ij} = \vec{a_i}\cdot\vec{a_j} = 0$ for $i \neq j$

i.e. for 2D case $(i,j = 1,2)$ $g_{12} = 0,\ g_{21} = 0$:.

Let us write ds^2 for 3D case

$$ds^2 = g_{11}(du_1)^2 + g_{22}(du_2)^2 + g_{33}(du_3)^2 \quad \text{—— (2)}$$

We can write a matrix $[g]$ for an orthogonal coordinate system as

$$[g] \equiv \begin{bmatrix} g_{11} & & 0 \\ & g_{22} & \\ 0 & & g_{33} \end{bmatrix}$$

Let us define (from 2)

$$ds_1 = \sqrt{g_{11}}\, du_1$$
$$ds_2 = \sqrt{g_{22}}\, du_2$$
$$ds_3 = \sqrt{g_{33}}\, du_3$$

or $ds_1 = h_1 du_1$, $ds_2 = h_2 du_2$ and $ds_3 = h_3 du_3$

where h_1, h_2 and h_3 are scale factors

Now we can write (2) as

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2 \quad \text{—— (3)}$$

Note: for cartesian coordinate system -

$$g_{11} = g_{22} = g_{33} = 1$$

and $\boxed{ds^2 = du_1^2 + du_2^2 + du_3^2}$

Since we know that $h_1 = \sqrt{g_{11}}$, $h_2 = \sqrt{g_{22}}$ & $h_3 = \sqrt{g_{33}}$,

we can calculate h_1, h_2 & h_3 using expression for g_{ii}

Next we show how to calculate h:

We can write ds^2 as

$$ds^2 = \sum_{k=1}^{3} dx_k \, dx_k$$

or $$ds^2 = \sum_{k=1}^{3} \left[\left(\sum_{i=1}^{3} \frac{\partial x_k}{\partial u_i} du_i \right) \left(\sum_{j=1}^{3} \frac{\partial x_k}{\partial u_j} du_j \right) \right]$$

or $$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\sum_{k=1}^{3} \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j} \right) du_i \, du_j \quad \text{——— (4)}$$

~~Since~~ In previous lecture notes we have seen that

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} \, du_i \, du_j$$

Thus from (4)

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j}$$

or for orthogonal curvilinear coordinates

$$\boxed{g_{ii} = \sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial u_i} \right)^2} \quad \text{——— (5)}$$