05.02.2024

Q. Solve $(D^2+a^2)y = \sec ax$.

Soln: Questions of such type with RHS $\sec ax$, $\tan ax$, $\operatorname{cosec} ax$ are very important. Students must practice such questions many times.

For CF, $D^2+a^2=0 \Rightarrow D = \pm ai$

$\therefore$ CF $= c_1 \cos ax + c_2 \sin ax$.

For PI

$$PI = \frac{1}{D^2+a^2}\sec ax$$

$$\Rightarrow PI = \frac{1}{(D+ai)(D-ai)}\sec ax$$

Now we write $\sec ax = \frac{e^{aix}\cdot e^{-aix}}{\cos ax}$

$$\Rightarrow PI = \frac{(D+ai)-(D-ai)}{2ai(D+ai)(D-ai)}\sec ax$$

$$\Rightarrow PI = \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\sec ax$$

$$\Rightarrow PI = \frac{1}{2ai}\cdot\frac{1}{(D-ai)}\frac{e^{aix}\cdot e^{-aix}}{\cos ax} - \frac{1}{2ai(D+ai)}\frac{e^{-aix}\cdot e^{aix}}{\cos ax}$$

[Very important step]

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$$\therefore PI = \frac{1}{2ai(D-ai)} \frac{e^{aix} \cdot e^{-aix}}{\cos ax}$$

$$- \frac{1}{2ai(D+ai)} \frac{e^{-aix} \cdot e^{aix}}{\cos ax} \quad \text{— (A)}$$

Now Ist part of eq (A)

$$\text{PI of Ist part} = \frac{1}{2ai(D-ai)} e^{aix} \cdot \frac{e^{-aix}}{\cos ax}$$

Here we will integrate e^{aix}.

For this, we shall replace D by D + ai (where ai is e^{aix})

$$= \frac{e^{aix}}{2ai} \frac{1}{(D+\cancel{ai})-\cancel{ai}} \frac{e^{-aix}}{\cos ax}$$

$$= \frac{e^{aix}}{2ai} \frac{1}{D} \frac{e^{-aix}}{\cos ax}$$

$$= \frac{e^{aix}}{2ai} \int \frac{e^{-aix}}{\cos ax} dx$$

$$\Rightarrow PI_{\text{of 1st part}} = \frac{e^{aix}}{2ai}\int \frac{\cos ax - i\sin ax}{\cos ax}\,dx$$

$$\Rightarrow PI_{\text{of 1st part}} = \frac{e^{aix}}{2ai}\int \left[1 - i\tan ax\right] dx$$

$$\Rightarrow PI_{\text{of 1st part}} = \frac{e^{aix}}{2ai}\left[x + \frac{i}{a}\log\cos ax\right] \quad \text{(B)}$$

Again

$$PI \text{ of 2nd part} = \frac{1}{2ai(D+ai)}\, e^{-aix} \cdot \frac{e^{aix}}{\cos ax}$$

$$= \frac{e^{-aix}}{2ai}\,\frac{1}{\left[(D-ai)+ai\right]}\,\frac{e^{aix}}{\cos ax}$$

$$= \frac{e^{-aix}}{2ai}\,\frac{1}{D}\,\frac{e^{aix}}{\cos ax} = \frac{e^{-aix}}{2ai}\int \frac{e^{aix}}{\cos ax}\,dx$$

$$= \frac{e^{-aix}}{2ai}\int \frac{\cos ax + i\sin ax}{\cos ax}\,dx$$

$$= \frac{e^{-aix}}{2ai}\int \left[1 + i\tan ax\right] dx$$

∴ PI of 2nd part

$$= \frac{e^{-aix}}{2ai}\left[x + \frac{i}{a}\log \sec ax\right]$$

$$= \frac{e^{-aix}}{2ai}\left[x - \frac{i}{a}\log \cos ax\right] \text{—(C)}$$

using eq (B) and eq (C) in (A)

we have

$$PI = \frac{e^{aix}}{2ai}\left[x + \frac{i}{a}\log \cos ax\right]$$

$$- \frac{e^{-aix}}{2ai}\left[x - \frac{i}{a}\log \cos ax\right]$$

$$\Rightarrow PI = \frac{x}{2ai}\left[e^{aix} - e^{-aix}\right]$$

$$+ \frac{\log \cos ax}{2a^2}\left[e^{aix} + e^{-aix}\right]$$

$$\Rightarrow PI = \frac{x}{2ai} \times 2i \sin ax$$

$$+ \frac{\log \cos ax}{2a^2} \times 2\cos ax$$

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$$\Rightarrow PI = \frac{x}{a}\sin ax + \frac{\cos ax}{a^2}\log\cos ax.$$

Hence the complete solution is given by

$$y = CF + PI$$

$$\Rightarrow y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{\cos ax}{a^2}\log\cos ax.$$