

06/02/2024

~~very very important~~

Q Solve $\frac{d^2y}{dx^2} + a^2y = \operatorname{cosec} ax$ — (A)

Tough

Soln

For CF, $\frac{d^2}{dx^2} + a^2 = 0$

$\Rightarrow D^2 + a^2 = 0 \Rightarrow D = \pm ai$

\therefore CF = $c_1 \operatorname{cosec} ax + c_2 \sin ax$ — (B)

For PI

PI = $\frac{1}{D^2 + a^2} \operatorname{cosec} ax$

\Rightarrow PI = $\frac{(D+ai) - (D-ai)}{2ai(D+ai)(D-ai)} \operatorname{cosec} ax$

\Rightarrow PI = $\frac{1}{2ai} \left[\frac{1}{D-ai} - \frac{1}{D+ai} \right] \frac{1}{\sin ax}$

\Rightarrow PI = $\frac{1}{2ai} \cdot \frac{1}{D-ai} \cdot \frac{1}{\sin ax} - \frac{1}{2ai} \cdot \frac{1}{(D+ai)} \cdot \frac{1}{\sin ax}$

\Rightarrow PI = $\frac{1}{2ai(D-ai)} \cdot e^{aix} \cdot \frac{1}{\sin ax} - \frac{1}{2ai(D+ai)} \cdot e^{-aix} \cdot \frac{1}{\sin ax}$

$\frac{1}{2ai(D+ai)} \cdot e^{-aix} \cdot \frac{1}{\sin ax}$

$$\Rightarrow PI = \frac{e^{aix}}{2ai} [(D+ai) - ai] \frac{e^{-aix}}{\sin ax}$$

$$- \frac{e^{-aix}}{2ai} [(D-ai) + ai] \frac{e^{aix}}{\sin ax}$$

$$PI = \frac{e^{aix}}{2ai} \frac{1}{D} \frac{e^{-aix}}{\sin ax} - \frac{e^{-aix}}{2ai} \frac{1}{D} \frac{e^{aix}}{\sin ax}$$

$$2) PI = \frac{e^{aix}}{2ai} \int \frac{\cos ax - i \sin ax}{\sin ax} dx$$

$$- \frac{e^{-aix}}{2ai} \int \frac{\cos ax + i \sin ax}{\sin ax} dx$$

$$3) PI = \frac{e^{aix}}{2ai} \int [\cot ax - i] dx$$

$$- \frac{e^{-aix}}{2ai} \int [\cot ax + i] dx$$

$$\Rightarrow PI = \frac{e^{aix}}{2ai} \left[\frac{1}{a} \log \sin ax - ix \right]$$

$$- \frac{e^{-aix}}{2ai} \left[\frac{1}{a} \log \sin ax + ix \right]$$

$$\Rightarrow PI = \frac{1}{2a^2 i} \log \sin ax \left(\frac{e^{aix} - e^{-aix}}{e^{aix} + e^{-aix}} \right)$$

$$- \frac{x}{2a} \left(\frac{e^{aix} - e^{-aix}}{e^{aix} + e^{-aix}} \right)$$

$$\Rightarrow PI = \frac{1}{2a^2 i} \log \sin ax \times \cancel{\frac{e^{aix} - e^{-aix}}{e^{aix} + e^{-aix}}}$$

$$- \frac{x}{2a} \times \cancel{\frac{e^{aix} - e^{-aix}}{e^{aix} + e^{-aix}}} \times \cos ax$$

$$\Rightarrow PI = \frac{\sin ax \log \sin ax}{a^2} - \frac{x}{a} \cos ax$$

Hence the given solution is given by

$$y = CF + PI$$

where CF and PI are

given by eq (B) and (C).