07/02/2024

Vector

Important Formulae:

(1) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

(2) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

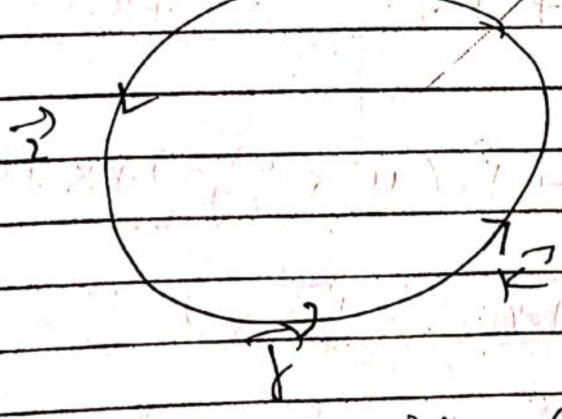
(3) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$

(4) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$

(5) $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(6)



$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

Also $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

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Theorem Prove that

$$\operatorname{div}(\vec{U}\times\vec{V}) = \vec{V}\cdot \operatorname{curl}\vec{u} - \vec{u}\cdot \operatorname{curl}\vec{V}$$

or $$\nabla\cdot(\vec{U}\times\vec{V}) = \vec{V}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{V})$$

Proof:→ $$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\therefore\ \nabla\cdot(\vec{U}\times\vec{V}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{U}\times\vec{V})$$

$$= \vec{i}\cdot\frac{\partial}{\partial x}(\vec{U}\times\vec{V}) + \vec{j}\cdot\frac{\partial}{\partial y}(\vec{U}\times\vec{V}) + \vec{k}\cdot\frac{\partial}{\partial z}(\vec{U}\times\vec{V})$$

$$= \vec{i}\cdot\left(\frac{\partial\vec{u}}{\partial x}\times\vec{V} + \vec{U}\times\frac{\partial\vec{V}}{\partial x}\right)$$
$$+ \vec{j}\cdot\left(\frac{\partial\vec{u}}{\partial y}\times\vec{V} + \vec{u}\times\frac{\partial\vec{V}}{\partial y}\right)$$
$$+ \vec{k}\cdot\left(\frac{\partial\vec{u}}{\partial z}\times\vec{V} + \vec{u}\times\frac{\partial\vec{V}}{\partial z}\right)$$

$$\nabla\cdot(\vec{U}\times\vec{V}) = \left[\vec{i}\cdot\left(\frac{\partial\vec{u}}{\partial x}\times\vec{V}\right) + \vec{j}\cdot\left(\frac{\partial\vec{u}}{\partial y}\times\vec{V}\right) + \vec{k}\cdot\left(\frac{\partial\vec{u}}{\partial z}\times\vec{V}\right)\right]$$
$$+ \left[\vec{i}\cdot\left(\vec{u}\times\frac{\partial\vec{V}}{\partial x}\right) + \vec{j}\cdot\left(\vec{u}\times\frac{\partial\vec{V}}{\partial y}\right) + \vec{k}\cdot\left(\vec{u}\times\frac{\partial\vec{V}}{\partial z}\right)\right] \quad \text{---(A)}$$

For 1st part of (A) $\vec{i}\cdot\left(\frac{\partial\vec{u}}{\partial x}\times\vec{V}\right) = \left(\vec{i}\times\frac{\partial\vec{u}}{\partial x}\right)\cdot\vec{V}$

Because $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

$\therefore$ 1st part of (A) can be rewritten as

$$= \left(\vec{i} \times \frac{\partial \vec{u}}{\partial x}\right) \cdot \vec{v} + \left(\vec{j} \times \frac{\partial \vec{u}}{\partial y}\right) \cdot \vec{v} + \left(\vec{k} \times \frac{\partial \vec{u}}{\partial z}\right) \cdot \vec{v}$$

$$= \left[\vec{i} \times \frac{\partial \vec{u}}{\partial x} + \vec{j} \times \frac{\partial \vec{u}}{\partial y} + \vec{k} \times \frac{\partial \vec{u}}{\partial z}\right] \cdot \vec{v}$$

$$= \left[\left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}\right) \times \vec{u}\right] \cdot \vec{v}$$

$$= (\nabla \times \vec{u}) \cdot \vec{v} \qquad \text{(B)}$$

$$= \vec{v} \cdot (\nabla \times \vec{u})$$

2nd part of eq (A) can be rewritten as

$$= \vec{i} \cdot \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x}\right) + \vec{j} \cdot \left(\vec{u} \times \frac{\partial \vec{v}}{\partial y}\right) + \vec{k} \cdot \left(\vec{u} \times \frac{\partial \vec{v}}{\partial z}\right)$$

$$\cancel{= \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x}\right) \cdot \vec{i} + \left(\vec{u} \times \frac{\partial \vec{v}}{\partial y}\right) \cdot \vec{j} + \left(\vec{u} \times \frac{\partial \vec{v}}{\partial z}\right) \cdot \vec{k}}$$

$$= -\left[\vec{i} \cdot \left(\frac{\partial \vec{v}}{\partial x} \times \vec{u}\right) + \vec{j} \cdot \left(\frac{\partial \vec{v}}{\partial y} \times \vec{u}\right) + \vec{k} \cdot \left(\frac{\partial \vec{v}}{\partial z} \times \vec{u}\right)\right]$$

$$= -\left[\left(\vec{i} \times \frac{\partial \vec{v}}{\partial x}\right) \cdot \vec{u} + \left(\vec{j} \times \frac{\partial \vec{v}}{\partial y}\right) \cdot \vec{u} + \left(\vec{k} \times \frac{\partial \vec{v}}{\partial z}\right) \cdot \vec{u}\right]$$

$$= -\left[\vec{i} \times \frac{\partial \vec{v}}{\partial x} + \vec{j} \times \frac{\partial \vec{v}}{\partial y} + \vec{k} \times \frac{\partial \vec{v}}{\partial z}\right] \cdot \vec{u}$$

$$= -\left[\left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}\right) \times \vec{v}\right] \cdot \vec{u}$$

$$= -(\nabla \times \vec{v}) \cdot \vec{u} = -\vec{u} \cdot (\nabla \times \vec{v}) \qquad \text{(C)}$$

S[illegible]y eq (B) and (C) in (A), we have

$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})$ proved