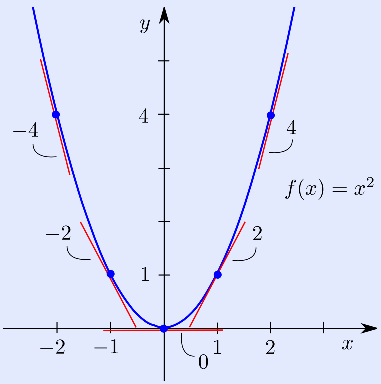
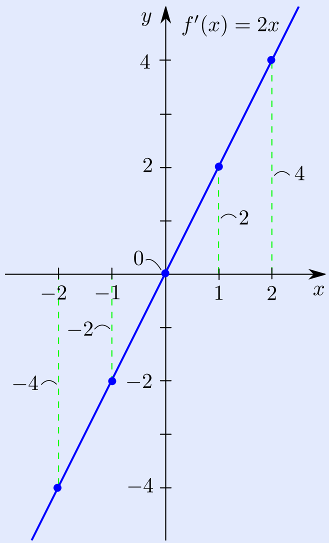
15. Graph of derivative

15.1. Two ways to interpret derivative

The function $f(x) = x^2$ has derivative $f'(x) = 2x$. This derivative is a general slope function. It gives the slope of any line tangent to the graph of f. For instance, if we want the slope of the tangent line at the point $(-2, 4)$, we evaluate the derivative at the x-coordinate of this point and get $f'(-2) = -4$. A few tangent lines are shown in the figure on the left, each tagged with its slope.





The derivative $f'(x) = 2x$ has a second interpretation. We can forget about the original function f and view f' as a function in its own right. The graph of f' is pictured on the right above. As always, the height of the graph above a number is given by the function evaluated at the number. For instance, the height of the graph of f' above -2 is $f'(-2) = -4$ (so the graph is actually below the number due to the negative sign).

Taking $x = -2$ as an example, we have seen two ways to interpret $f'(-2)$ (which equals -4). On the one hand, it is the slope of the line tangent to the graph of the original function f above -2. On the other hand, it is the height of the graph of the derivative f' above -2. This illustrates a general principle:

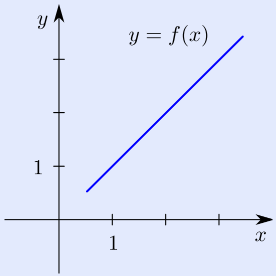
> At any number a,
>
> *slope* of the graph of f at a $=$ *height* of the graph of f' at a
>
> Both of these quantities equal $f'(a)$.

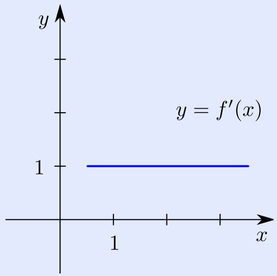
(The phrase "slope of the graph of f at a" is short for "slope of the line tangent to the graph of f at the point $(a, f(a))$.")

15.2. Relating graph of function to graph of derivative

We give a series of examples with the graph of a function on the left and the graph of its derivative on the right, each followed by an explanation.

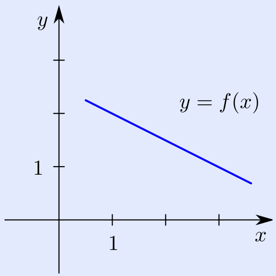
15.2.1 Example

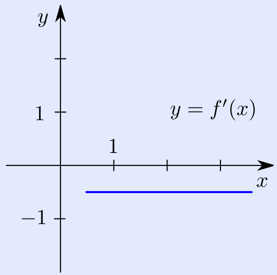




At each x, the graph of f has slope 1, so at each x the height of the graph of f' is 1 as well.

15.2.2 Example

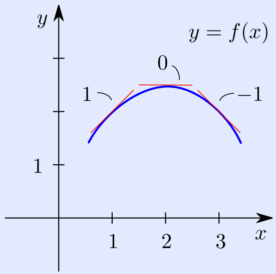


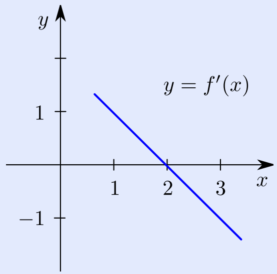


At each x, the graph of f has slope $-1/2$, so at each x the height of the graph of f' is $-1/2$

as well.

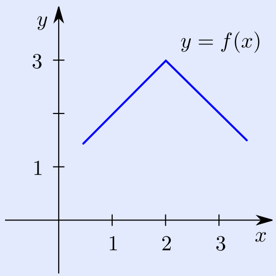
15.2.3 Example

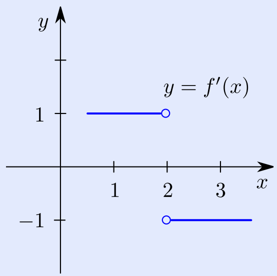




As indicated, the graph of f has slope 1 at $x = 1$, slope 0 at $x = 2$, and slope -1 at $x = 3$. These slopes are the heights of the graph of f' at $x = 1$, $x = 2$, and $x = 3$, respectively.

15.2.4 Example





To the left of 2 the graph of f has slope 1, so, to the left of 2 the graph of f' has height 1. Similarly, to the right of 2 the graph of f has slope -1, so, to the right of 2 the graph of f' has height -1. This leaves the behavior of f' right at 2 to be determined. According to the definition of the derivative,

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \tag{1}$$

provided the limit on the right-hand side of this equation exists. If the limit does not exist, then $f'(2)$ is undefined. We show that this latter is the case by showing that the one-sided limits are not the same. First, writing the equations of the two lines that make up the graph of f we get

$$f(x) = \begin{cases} x+1, & x \leq 2, \\ 5-x, & x \geq 2. \end{cases}$$

Therefore,

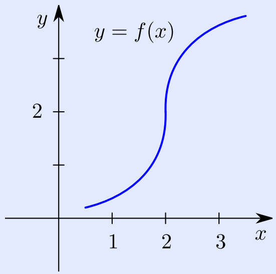
$$\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{((2+h)+1) - 3}{h} = 1,$$

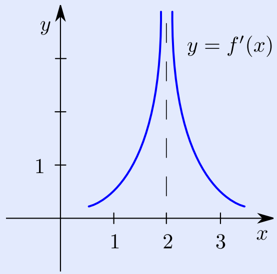
while

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{(5-(2+h))-3}{h} = -1,$$

so the two-sided limit in (1) does not exist. We conclude that $f'(2)$ is undefined and so we leave holes in the graph of f' at 2.

15.2.5 Example





The slope of the graph of f is $1/2$ at 1 and it gets ever greater as x approaches 2 from the left, so the height of the graph of f' is $1/2$ at 1 and it gets ever greater as x approaches 2 from the left. The behavior of both graphs to the right of 2 is similar, but reversed. In this case, the two-sided limit in (1) is ∞. In particular, the limit does not exist so that $f'(2)$ is undefined.

15.2.6 Example

The graph of f has slope -1 to the left of 2 and slope 2 to the right of 2, so the graph of f' has height -1 to the left of 2 and height 2 to the right of 2. For $x < 2$, $f(x) = 4 - x$, so

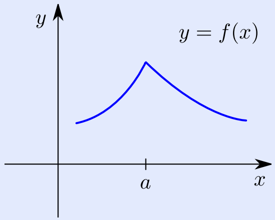
$$\begin{aligned}\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-} \frac{(4-(2+h))-1}{h} \\ &= \lim_{h\to 0^-} \frac{1-h}{h} \qquad \left(\frac{\text{about } 1}{\text{small neg.}}\right) \\ &= -\infty.\end{aligned}$$

Therefore, the limit in (1) does not exist and $f'(2)$ is undefined.

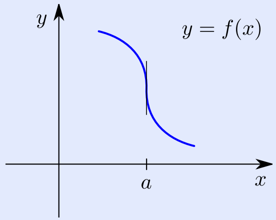
15.3. Where the derivative is undefined

The last three examples in the previous section illustrate the three main ways a derivative can be undefined at a number. These ways are summarized in the following statement.

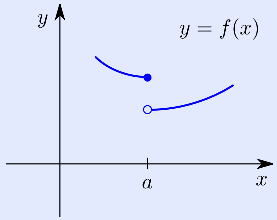
Where the derivative is undefined. The derivative f' of a function f is undefined at any number a for which $f(a)$ is undefined and also at any a for which the graph of f satisfies one of the following:



corner at a

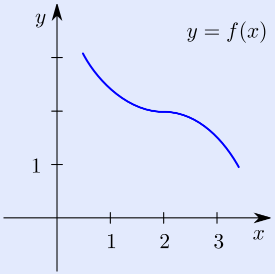


vertical tangent at a

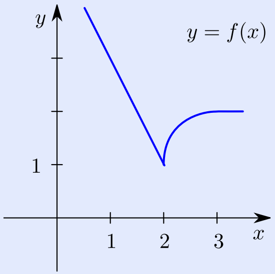


break at a
(f discontinuous at a)

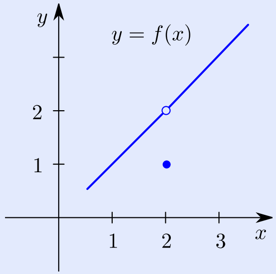
15 – 1 Sketch the graph of the derivative f' of the function f having the pictured graph:



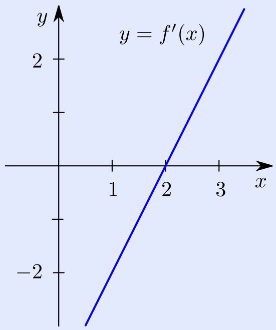
15 – 2 Sketch the graph of the derivative f' of the function f having the pictured graph:



15 − 3 Give an argument similar to that following 15.2.6 to show that the derivative $f'(2)$ does not exist for the function f with graph as pictured:



15 − 4 Pictured is the graph of the *derivative* f' of an unknown function f:



(a) Sketch the graph of f given that $f(1) = 1$.

(b) Sketch the graph of f given that $f(1) = 2$.

(c) Describe the graphs of all the possibilities for the function f.