① U.G MJC - Math
Trigonometry

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① State & prove Gregory's Series.

Statement - If $-\frac{1}{4}\pi \le \theta \le \frac{1}{4}\pi$ then

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - - - (-1)^{n-1}\frac{\tan^{2n-1}\theta}{2n-1} + - - \text{to } \infty$$

Proof:- We know that

$$e^{i\theta} = \cos\theta + i\sin\theta = \cos\theta(1 + i\tan\theta)$$

From definition of logarithm,

$$i\theta = \log\{\cos\theta(1 + i\tan\theta)\}$$
$$= \log\cos\theta + \log(1 + i\tan\theta)$$

$\because -\frac{1}{4}\pi \le \theta \le \frac{1}{4}\pi$

$\therefore -1 \le \tan\theta \le 1$

$$\therefore i\theta = \log\cos\theta + i\tan\theta - \frac{1}{2}i^2\tan^2\theta + \frac{1}{3}i^3\tan^3\theta - \frac{1}{4}i^4\tan^4\theta + - - -$$

$$= \left(\log\cos\theta + \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + - - -\right) + i\left(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + - -\right)$$

Equating the imaginary parts on both sides,

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - - - - + (-1)^{n-1}\frac{\tan^{2n-1}\theta}{2n-1} + - - \text{to } \infty$$

where, $-\frac{1}{4}\pi \le \theta \le \frac{1}{4}\pi$

This is called Gregory's Series.

proved.

Evaluation of π

The value of π can be obtained as a sum of a series by using different forms of Gregory series.

Putting $\theta = \frac{\pi}{4}$ in eqn (i), we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \text{ to } \infty \quad \left[\because \tan\frac{1}{4}\pi = 1\right]$$

$$= \frac{2}{1\cdot3} + \frac{2}{5\cdot7} + \frac{2}{9\cdot11} + \cdots \text{ to } \infty$$

$$\therefore \frac{\pi}{8} = \frac{1}{1\cdot3} + \frac{1}{5\cdot7} + \frac{1}{9\cdot11} + \cdots \text{ to } \infty$$

It can very easily be proved that

$$\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} + \cdots$$

Putting $x = \frac{1}{2}$ and $x = \frac{1}{3}$ in the expansion of $\tan^{-1}x$, we find the Euler's series as follows:

$$\frac{\pi}{4} = \left[\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5} - \cdots \text{ to } \infty\right]$$

$$+ \left[\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} - \cdots \text{ to } \infty\right]$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) \cdots \text{ to } \infty$$