B. Sc. (Hons.) Part II Paper III

[redacted]

Infinite series

* Comparison test

If ΣU_n and ΣV_n be two series of (+)ve terms then

$\underset{n\to\infty}{Lt} \frac{U_n}{V_n}$ be finite and non-zero,

the series will be both convergent or both convergent.

* The infinite series

$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ to ∞ is

convergent if $p > 1$ and
divergent if $p \leq 1$.

Working rule

1. Find the n^{th} term U_n.
2. consider another auxiliary series whose n^{th} term (V_n) is equal to

$$V_n = \frac{\text{term of highest power of } n \text{ in numerator of the}}{\text{term of highest power of } n \text{ in the denominator of } U_n}.$$

3. Find $\frac{U_n}{V_n}$.

4. Find $\underset{n \to \infty}{Lt} \frac{U_n}{V_n}$.

If $\underset{n \to \infty}{Lt} \frac{U_n}{V_n}$ = finite and non-zero

then proceed as following.

5. compare V_n with the series $\sum \frac{1}{n^p}$.

$\sum \frac{1}{n^p}$ ~~which~~ is cgt if $p > 1$

is dgt. if $p \leq 1$.

6. By comparison test both the series $\sum U_n$ and $\sum V_n$ will converge or diverge simultaneously

EXAMPLES

1. Test the convergence of the series

$$1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \cdots + \frac{1+n}{1+n^2} + \cdots \text{to } \infty$$

Soln The series can be written as

$$\frac{1+1}{1+1^1} + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \cdots + \frac{1+n}{1+n^2} + \cdots \text{to } \infty$$

Then its n^{th} term $(=U_n) = \dfrac{1+n}{1+n^2}$

Let us consider another series whose n^{th} term is V_n and $V_n = \dfrac{n}{n^2} = \dfrac{1}{n}$

$$\therefore \frac{U_n}{V_n} = \frac{\frac{1+n}{1+n^2}}{\frac{1}{n}} = \frac{n(1+n)}{1+n^2}$$

$$= \frac{n+n^2}{1+n^2} = \frac{n^2\left(\frac{1}{n}+1\right)}{n^2\left(\frac{1}{n^2}+1\right)}$$

$$= \frac{1+\frac{1}{n}}{1+\frac{1}{n^2}}$$

$$\therefore \underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{1+\frac{1}{n}}{1+\frac{1}{n^2}} = 1$$

which is finite and non-zero.

So, by comparison test, the two series $\sum U_n$ and $\sum V_n$ behave alike (i.e. converge or diverge simultaneously).

Now $V_n = \frac{1}{n} \Rightarrow \sum V_n = \sum \frac{1}{n}$

$\Rightarrow \sum V_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ to ∞

Comparing it with $\sum \frac{1}{n^p}$, we have $p = 1$.

$\Rightarrow \sum V_n$ is not convergent, i.e. divergent.

Hence, by comparison test, the given series $\sum U_n$ is dgt.

$\boxed{Q}$ Test the convergency of the series

$$\sum \frac{5n-3}{2n^3-1}.$$

Soln Here, $U_n = \frac{5n-3}{2n^3-1}$

$\Rightarrow$ Consider another series $\sum V_n$ with

$$V_n = \frac{n}{n^3} = \frac{1}{n^2}$$

So $\frac{U_n}{V_n} = \frac{n^2(5n-3)}{2n^3-1} = \frac{\frac{n^2(5n-3)}{n^3}}{\frac{2n^3-1}{n^3}} = \frac{5-\frac{3}{n}}{2-\frac{1}{n^3}}$

$\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \left(\frac{5-\frac{3}{n}}{2-\frac{1}{n^3}}\right) = \frac{5}{2}$ which is finite and non-zero

So, by comparison test, they behave alike.

But $\sum V_n = \sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots$ is convergent

$\Rightarrow \sum U_n$ (the given series) is also convergent.