Differential and Integral Equation

Show that $u(x) = \cos 2x$ is a solution of the integral equation

$$u(x) = \cos x + 3\int_0^{\pi} K(x,t)\, u(t)\, dt$$

where $K(x,t) = \begin{cases} \sin x \cos t \ ; & 0 \leq x \leq t \\ \cos x \cdot \sin t \ ; & t \leq x \leq \pi \end{cases}$

Soln — The given integral equation is

$$u(x) = \cos x + 3\int_0^{\pi} K(x,t)\, u(t)\, dt \quad \text{—} (1)$$

where $K(x,t) = \begin{cases} \sin x \cos t \ ; & 0 \leq x \leq t \\ \cos x \sin t \ ; & t \leq x \leq \pi \end{cases}$

Also given $u(x) = \cos 2x$

$$\Rightarrow u(t) = \cos 2t$$

Then RHS of equation (1) is

$$= \cos x + 3\left[\int_0^{x} k(x,t)\, u(t)\, dt + \int_0^{\pi} k(x,t)\, u(t)\, dt\right]$$

$$= \cos x + 3\left[\int_0^{x} \cos x \sin t \cos 2t\, dt + \int_0^{\pi} \sin x \cos t \cos 2t\, dt\right]$$

$$= \cos x + 3\cos x \int_0^{x} \cos 2t \sin t\, dt + 3\sin \int_0^{\pi} \cos 2t \cdot \cos t\, dt.$$

$$= \cos x + \frac{3}{2}\cos x \int_0^x (\sin 3t - \sin t)\, dt$$

$$+ \frac{3}{2}\sin x \int_0^{\pi} (\cos 3t + \cos t)\, dt$$

$$= \cos x + \frac{3}{2}\cos x \left[-\frac{1}{3}\cos 3t + \cos t \right]_0^x$$

$$+ \frac{3}{2}\sin x \left[\frac{1}{3}\sin 3t + \sin t \right]_0^{\pi}$$

$$= \cos x + \frac{3}{2}\cos x \left[-\frac{1}{3}\cos 3x + \cos x + \frac{1}{3} - 1 \right]$$

$$+ \frac{3}{2}\sin x \left[-\frac{1}{3}\sin 3x - \sin x \right]$$

$$= \cos x - \frac{1}{2}(\cos 3x \cos x + \sin 3x \sin x)$$

$$+ \frac{3}{2}(\cos^2 x - \sin x) - \cos x$$

$$= -\frac{1}{2}\cos(3x - x) + \frac{3}{2}\cos 2x$$

$$= -\frac{1}{2}\cos 2x + \frac{3}{2}\cos 2x$$

$$= \cos 2x = u(x)$$

Hence, $u(x) = \cos 2x$ is the solution of given integral equation.