P.G. Semester - II, paper - CC-VII

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Differential and Integral Equation

Q- Reduce the following boundary value problem into an integral equation

$$u''(x) + \lambda u = 0$$

with boundary conditions

$$u(0) = 0,\ u(l) = 0$$

Solⁿ: The given equation is

$$u''(x) + \lambda u = 0 \quad \text{——— (1)}$$

with boundary conditions

$$u(0) = 0,\ u(l) = 0 \quad \text{——— (2)}$$

Equation (1) can be written as

$$u''(x) = -\lambda u(x) \quad \text{——— (3)}$$

Integrating both sides (3) w.r.to x from 0 to x, we get

$$\int_0^x u''(x)\,dx = -\lambda \int_0^x u(x)\,dx$$

$$\Rightarrow \left[u'(x)\right]_0^x = -\lambda \int_0^x u(x)\,dx$$

$$\Rightarrow u'(x) - u'(0) = -\lambda \int_0^x u(x)\,dx \quad \text{——— (4)}$$

Let $u'(0) = C$, where C is a constant ——— (5)

Then using (5) in (4), we get

$$u'(x) = C - \lambda \int_0^x u(x)\,dx \quad \text{--- (6)}$$

Integrating both sides of (6) w.r.to x, we get

$$\int_0^x u'(x)\,dx = C\int_0^x dx - \lambda \int_0^x u(x)\,dx^2$$

$$\Rightarrow \left[u(x)\right]_0^x = Cx - \lambda \int_0^x u(t)\,dt^2$$

$$\Rightarrow u(x) - u(0) = Cx - \lambda \int_0^x (x-t)\,u(t)\,dt$$

$$\Rightarrow u(x) - 0 = Cx - \lambda \int_0^x (x-t)\,u(t)\,dt \quad \text{(7)}$$

Now putting $x = l$ in equation (7), we get

$$u(l) = Cl - \lambda \int_0^l (l-t)\,u(t)\,dt$$

$$\Rightarrow 0 = Cl - \lambda \int_0^l (l-t)\,u(t)\,dt$$

$$\Rightarrow C = \frac{\lambda}{l} \int_0^l (l-t)\,u(t)\,dt \quad \text{--- (8)}$$

putting this value in equation (7), we get.

$$u(x) = \frac{\lambda}{l} x \int_0^l (l-t)\, u(t)\, dt - \lambda \int_0^l (x-t)\, u(t)\, dt \quad -(9)$$

$$\Rightarrow u(x) = \int_0^l \frac{\lambda x(l-t)}{l} u(t)\, dt - \lambda \int_0^l (x-t)\, u(t)\, dt$$

$$\Rightarrow u(x) = \int_0^x \frac{\lambda x(l-t)}{l} u(t)\, dt + \int_x^l \frac{\lambda x(l-t)}{l} u(t)\, dt - \int_0^x \lambda(x-t)\, u(t)\, dt$$

$$= \lambda \int_0^x \left[\frac{x(l-t)}{l} - (x-t)\right] u(t)\, dt + \lambda \int_0^l \frac{x(l-t)}{l} u(t)\, dt$$

$$= \lambda \int_0^x \frac{x(l-t) - l(x-t)}{l} u(t)\, dt + \lambda \int_0^l \frac{x(l-t)}{l} u(t)\, dt$$

$$= \lambda \left[\int_0^x \frac{t \cdot (l-x)}{l} u(t)\, dt + \int_x^l \frac{x(l-t)}{l} u(t)\, dt\right]$$

$$= \lambda \int_x^l K(x,t)\, u(t)\, dt$$

where $K(x,t) = \begin{cases} \dfrac{t(l-x)}{l} & \text{if } 0 < t < x \\ \dfrac{x(l-t)}{l} & \text{if } x < t < l \end{cases}$