2.2. "NORMED SPACE. BANACH SPACE":

NORM :-

Let X be a vector space over a field $\mathbb{K}$. A norm on a vector space X (real or complex) is a real valued function on X. $\|\cdot\|: X \longrightarrow \mathbb{R}$ whose value at an $x \in X$ is denoted by $\|x\|$ and which satisfies the following properties :-

(i) $\|x\| \geq 0$ (non-negativity)

(ii) $\|x\| = 0 \Leftrightarrow x = 0$

(iii) $\|\alpha x\| = |\alpha| \|x\|$ (homogenity of norm)

(iv) $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality)

for all $x, y \in X$ and $\alpha \in \mathbb{K}$.

* A norm on X defines a metric d on X which is given by
$$d(x,y) = \|x-y\| \ ; \ x, y \in X,$$
and is called the metric induced (or generated) by the norm

NORMED SPACE :-

A vector space X with a norm $\|\cdot\|$ defined on it is called normed space and is denoted by ~~X,d~~ $(X, \|\cdot\|)$ or simply X. It is also called a normed vector space or normed linear space.

→ A complete normed space (complete in the metric defined by the norm) is called the BANACH SPACE

* Every normed space and Banach space is a metric space but converse may not hold.

* A normed space is said to be real or complex depending on the field.

* The norm is continuous. that is $x \mapsto \|x\|$ is a continuous mapping of $(X, \|\cdot\|)$ into $\mathbb{R}$.

Examples :

(1) Let $X = \mathbb{R}$.
Define $\|\cdot\| : X \longrightarrow \mathbb{R}$ as
$\|x\| = |x| \quad \forall x \in X$.
$(\mathbb{R}, \|\cdot\|)$ is normed space (and a Banach space also).

(2) Euclidean space $\mathbb{R}^n$.
Let $X = \mathbb{R}^n$.
Define $\|\cdot\|_2 : X \longrightarrow \mathbb{R}$ by
$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \quad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.
$(\mathbb{R}^n, \|\cdot\|_2)$ is a Banach space.

(3) Unitary space $\mathbb{C}^n$.
Let $X = \mathbb{C}^n$.
Define $\|\cdot\|_2 : X \longrightarrow \mathbb{R}$ by
$\|x\|_2 = \left(\sum_{i=1}^{n} |\xi_i|^2\right)^{\frac{1}{2}} \quad \forall x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$
$(\mathbb{C}^n, \|\cdot\|_2)$ is a Banach space.

4(a). Define $\|\cdot\|_\infty : \mathbb{R}^n \longrightarrow \mathbb{R}$ by
$\|x\|_\infty = \sup_{1 \le i \le n} |x_i| \quad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

OR,

$\|x\|_\infty = \max_{1 \le i \le n} |x_i| \quad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

This is called the sup-norm (or the uniform norm)