$\beta \subseteq \mathcal{J}$

Reason:- let $B \in \beta$ T.S:- $B \in \mathcal{J}$: let $x \in B$, then $\exists$ $B \in \beta$ s.t $x \in B \subseteq B$

✱ Basis for a Topology:—

Def^n: If X is a set, a basis for a Topo. on X is a collection β of subset of X (called basis elts) such that

(i) for each $x \in X$, $\exists$ at least one basis element B (i.e $B \in \beta$) s.t $x \in B$; ($B \subseteq X$)

(ii) If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \beta$. then $\exists$ $B_3 \in \beta$ s.t $x \in B_3 \subset B_1 \cap B_2$.

→ If cond^n (i) & (ii) satisfied by β, then we define Topo. $\mathcal{J}$ generated by β as $\mathcal{J} = \{U \subseteq X \mid \forall x \in U \; \exists B \in \beta \text{ s.t } x \in B \subset U\}$.

pg 79

EX(3) ~~If X is any set, the collection of all one-point subsets of X is a basis for the discrete Topo. on X.~~ OR EX(1) $X = \mathbb{R}$, $\beta = \{\{x\} \mid x \in \mathbb{R}\}$

Result:- $\mathcal{J} = \{U \subseteq X \mid \forall x \in U \; \exists B \in \beta : x \in B \subset U\}$ is Topo. on X

OR; let us now check that the collection $\mathcal{J}$ generated by the basis β is, infact a Topology on X.

the cond^n of openness vacuously → [a subset U of X is given by: if for each $x \in U$ $\exists$ $B \in \beta$ s.t $x \in B \subset U$ —(✱)]

Sol^n: (i) If $U = \phi$ then it satisfy (✱) the cond^n of openness vacuously.

i.e $\phi \in \mathcal{J}$

let $U = X$. let $x \in X$

then by definition of β $\exists$ $B \in \beta$ s.t. $x \in B \subset X$ $\Rightarrow$ X is open. i.e. $X \in \mathcal{J}$

(ii) now let us take an Indexed family $\{U_\alpha\}_{\alpha \in \Lambda}$ of element of $\mathcal{J}$.

let $U = \bigcup_{\alpha \in \Lambda} U_\alpha$.

13.2 ($\mathcal{C} = \{U \subseteq X \mid x \in U, \exists\, C \in \mathcal{C}$ s.t $x \in C \subset U\}$.
OR $\mathcal{T}'$

Remark: By above lemma, every open subset of X can be written as a union of basis elements. [ie $\bigcup_{B \in \mathcal{B}} B = X$]
This represents for open sets is not however unique.
* [$\bigcup_{U \in \mathcal{T}} U = X$ also]

~ Obtaining basis for a given Topo. ~

2009.
Lemma (13.2): Let X be a Topo. space. suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U there is an elt C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the Topo. of X.

Proof: We must show that $\mathcal{C}$ is a basis. The first condition for a basis is easy.

(i) Let $x \in X$.
Since X is itself open set (being a member of Topo). ie $X \in \mathcal{T}$.
$\therefore$ by definition of $\mathcal{C}$.
$\exists\, C \in \mathcal{C}$ s.t $x \in C \subset X$.

Now we check second:
(2) Let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathcal{C}$.
Since C_1 & C_2 are open {$\therefore C_1 \cap C_2$ is open ie $C_1 \cap C_2 \in \mathcal{T}$.
Since $C_1 \cap C_2$ is open.
Then by definition of $\mathcal{C}$.
$\exists\, C_3 \in \mathcal{C}$ s.t $x \in C_3 \subset C_1 \cap C_2$.
$\Rightarrow \mathcal{C}$ is a basis. #
Let $\mathcal{T}$ be the given Topo. on X. and $\mathcal{T}'$ be the Topo. generated by $\mathcal{C}$