Norm of $\mathbb{R}^n$

① 2-norm
② $l^1$-norm
③ p-norm.

② (b) Define $\|\cdot\|_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\|x\|_1 = \left(\sum_{i=1}^{n} |\xi_i|^1\right)^{\frac{1}{1}} \quad \forall\ x = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n.$$

$\|\cdot\|_1$ is called $l^1$-norm and $(\mathbb{R}^n, \|\cdot\|_1)$ is a Banach space.

③ (c) Define $\|\cdot\|_p : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\|x\|_p = \left(\sum_{i=1}^{n} |\xi_i|^p\right)^{\frac{1}{p}} \quad \forall\ x = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n.$$

for $1 \le p < \infty$, $\|\cdot\|_p$ is called the p-norm.

$\longrightarrow$ If $0 < p < 1$; then $\|\cdot\|_p$ is not a norm on $X = \mathbb{R}^n$.

Counter example.

Let $x = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$
$y = (0, 1, 0, \dots, 0) \in \mathbb{R}^n$.

then

$$\left.\begin{array}{l} \text{(i) } \|x\|_p \ge 0 \\ \text{(ii) } \|x\|_p = 0 \Leftrightarrow x = 0 \\ \text{(iii) } \|\alpha x\|_p = |\alpha| \|x\|_p \end{array}\right\} \text{satisfied.}$$

but (iv) $\|x\|_p = 1^p + 0^p + \dots + 0^p = 1$, $\|y\|_p = 0^p + 1^p + \dots + 0^p = 1$

& $\|x+y\|_p = (1^p + 1^p + 0^p + \dots + 0^p)^{1/p} = (2 \cdot 1^p)^{\frac{1}{p}} = 2^{1/p}$ $\because p < 1$ $\therefore \frac{1}{p} > 1$

so, we get, $\|x+y\|_p \nleq \|x\|_p + \|y\|_p$ if $0 < p < 1$ {triangle equality not hold.

$\|x+y\|_p > \|x\|_p + \|y\|_p$ if $0 < p < 1$.

$\Rightarrow \|\cdot\|_p$ is not a norm on $X$ if $0 < p < 1$.

(5) The sequence space $l^p$:-

Let $p \ge 1$ be fixed real number.

Each element in the space $l^p$ is a sequence $x = \langle \xi_i \rangle_{i=1}^{\infty} = \langle \xi_1, \xi_2, \dots \rangle$ of numbers s.t. the sereces $\sum_{i=1}^{\infty} |\xi_i|^p$ cgt. or $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$

i.e,

$$l^p = \left\{ x = \langle \xi_i \rangle_{i=1}^{\infty} \ \middle|\ \sum_{i=1}^{\infty} |\xi_i|^p < \infty,\ p \ge 1 \text{ fixed} \right\}.$$

$l^p$ is a vector space under pointwise addition and multiplication.

Define $\|\cdot\|_p : \ell^p \longrightarrow \mathbb{R}$ by

$$\|x\|_p = \left(\sum_{i\geq 1}^{\infty} |\xi_i|^p\right)^{\frac{1}{p}} \quad \forall\ x = \langle \xi_i \rangle_{i\geq 1}^{\infty} \in \ell^p$$

$1 \leq p < \infty$.

$(\ell^p, \|\cdot\|_p)$ is a normed space (& also a Banach sp.)

Eg: Let $x = \langle \xi_i \rangle_{i\geq 1}^{\infty} = \langle \frac{1}{n} \rangle_{n\geq 1}^{\infty} \in \ell^p$ for $p>1$

then $\sum_{i\geq 1}^{\infty} |\xi_i|^p = \sum_{i\geq 1}^{\infty} \left|\frac{1}{i^p}\right| < \infty$ by P-Test

viz says,

$\sum \frac{1}{n^p}$ cgs if $p>1$

$\sum$ " dgs if $p \leq 1$

$\ell^p$ for $p = 1, 2$

$$\ell^1 = \left\{ \langle \xi_i \rangle_{i\geq 1}^{\infty} ; \sum_{i\geq 1}^{\infty} |\xi_i| \text{ is } \cancel{\text{dgt}}\ \text{cgt} \right\}$$

$$\ell^2 = \left\{ \{\xi_i\}_{i\geq 1}^{\infty} ; \sum_{i\geq 1}^{\infty} |\xi_i|^2 \text{ is cgt} \right\}$$

→ Give an example of sequence which in in $\ell^p, p>1$ ie, in $\ell^2, \ell^3, \ldots\ldots, \ell^p, \ldots$ but not in $\ell^1$.

eg: Let $x = \langle \xi_i \rangle_{i\geq 1}^{\infty} = \langle \frac{1}{n} \rangle_{n\geq 1}^{\infty}$ ∴ $\sum_{i\geq 1}^{\infty} \left|\frac{1}{i}\right|^p = \sum \frac{1}{i^p}$, $\sum \frac{1}{i^p} < \infty$ for $p>1$.

$$\Rightarrow x = \langle \frac{1}{n} \rangle_{n\geq 1}^{\infty} \in \ell^p, \ p>1$$

but $x = \langle \xi_i \rangle_{i\geq 1}^{\infty} = \langle \frac{1}{n} \rangle_{n\geq 1}^{\infty}$ ∴ $\sum_{i\geq 1}^{\infty} \left|\frac{1}{i}\right|^p = \sum \frac{1}{i}$ dgt, $p=1$.

$$\Rightarrow x = \langle \frac{1}{n} \rangle_{n\geq 1}^{\infty} \notin \ell^1.$$

(6) $\ell^{\infty} = \left\{ \langle \xi_n \rangle_{n\geq 1}^{\infty} : \xi_n \in \mathbb{K} \ (= \mathbb{R})\ \forall\ n \in \mathbb{N},\ \langle \xi_n \rangle \text{ is bdd} \right\}$

i.e, $\ell^{\infty}$ is the set of all ~~real~~ bdd sequences of ~~numbers~~ numbers.

Define $\|\cdot\|_{\infty} : \ell^{\infty} \longrightarrow \mathbb{R}$ by

$$\|x\|_{\infty} = \sup_{1\leq j<\infty} |\xi_j| \quad \forall\ x \in \ell^{\infty} \ \left(= \langle \xi_j \rangle_{j\geq 1}^{\infty}\right).$$