In last note we have discussed gradient. For any scalar function $\phi(x,y,z)$, the grad ϕ or $\vec{\nabla}\phi$ is defined as

$$\vec{\nabla}\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

↑

vector differential operator

One can use $\hat{i}, \hat{j}, \hat{k}$ in place of $\hat{x}, \hat{y}, \hat{z}$

(A). Divergence: If we operate ∇ on a vector ~~$\vec{A} = A_x$~~ $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$. We dot $\vec{\nabla}$ into the $\vec{A}$ and obtain

$$\boxed{\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

$\nabla\cdot\vec{A} \rightarrow$ divergence of $\vec{A}$.
Scalar quantity

Some Examples:

(a) Evaluating $\nabla\cdot\vec{r}$, $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$$\nabla\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\boxed{\nabla\cdot\vec{r} = 3}$$

(b) ~~Calculate $\nabla^2\left(\frac{1}{r}\right)$~~. Prove $\nabla\cdot\nabla\phi = \nabla^2\phi$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

$$\cancel{\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right.}$$

$$\nabla\cdot\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi,$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

↑

We call it Laplacian operator.

H.W. Prove $\nabla^2\left(\frac{1}{r}\right) = 0$

Hint! $r = \sqrt{x^2+y^2+z^2}$

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$