

Solution of Laplace's equation in two dimensional cylindrical co-ordinates (r, θ) .

The Laplace equation

$\nabla^2 U = 0$ in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 0 \quad \text{--- (1)}$$

If we assume that the function U is independent of co-ordinate z , then Laplace's equation in two dimensional cylindrical coordinates takes the form.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} = 0 \quad \text{--- (2)}$$

In this equation U is the function of r and θ , therefore by the method of separation of variables U may be written as

$$U(r, \theta) = R(r) \Theta(\theta) \quad \text{--- (3)}$$

where R the function of r only and Θ is the function of θ only

Substituting this value of U in eqⁿ (2) we have

$$\Theta \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \frac{R}{r^2} \frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

Dividing this eqⁿ by $\frac{R\Theta}{r^2}$ we get

	JULY 2010			AUGUST		
5	12	19	26	Mon	30	2 9 16 23
6	13	20	27	Tue	31	3 10 17 24
7	14	21	28	Wed	4	11 18 25
8	15	22	29	Thu	5	12 19 26
9	16	23	30	Fri	6	13 20 27
10	17	24	31	Sat	7	14 21 28
11	18	25		Sun	1 8	15 22 29

$$\frac{1}{R} \left[r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} \right] = - \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} \quad \text{--- (4)}$$

In this equation left hand side is the function of r only; while right hand side is the function of θ only, therefore each side must be equal to the same constant n^2 (say)

i.e.
$$\frac{1}{R} \left[r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} \right] = n^2$$

or
$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} - n^2 R = 0 \quad \text{--- (5)}$$

and
$$- \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} = n^2$$

or
$$\frac{\partial^2 \Theta}{\partial \theta^2} + n^2 \Theta = 0 \quad \text{--- (6)}$$

The solution of equation (5) may be expressed as

$$R = A_n r^n + B_n r^{-n}, \quad n \neq 0 \quad \text{--- (7)}$$

while equation (6) represents simple harmonic motion and its solution may be expressed as

$$\Theta = C_n \cos n\theta + D_n \sin n\theta, \quad n \neq 0 \quad \text{--- (8)}$$

where A_n, B_n, C_n and D_n are arbitrary constants.

2010					MAY					2010					JUNE							
Mon	31	3	10	17	24	Mon	7	14	21	28	Mon	1	8	15	22	29	Mon	7	14	21	28	
Tue		4	11	18	25	Tue	1	8	15	22	29	Tue	2	9	16	23	30	Tue	8	15	22	29
Wed		5	12	19	26	Wed	2	9	16	23	30	Wed	3	10	17	24	Wed	9	16	23	30	
Thu		6	13	20	27	Thu	3	10	17	24	Thu	4	11	18	25	Thu	10	17	24	31		
Fri		7	14	21	28	Fri	4	11	18	25	Fri	5	12	19	26	Fri	11	18	25			
Sat		1	8	15	22	29	Sat	5	12	19	26	Sat	6	13	20	27	Sat	12	19	26		
Sun		2	9	16	23	30	Sun	6	13	20	27	Sun	7	14	21	28	Sun	13	20	27		

If $n = 0$, the equation (5) and (6) take the form

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} = 0 \quad \text{--- (9)}$$

$$\frac{\partial^2 \Theta}{\partial \theta^2} = 0 \quad \text{--- (10)}$$

The solution of these equation may be expressed as

$$R = A_0 \log r + B_0 \quad \text{--- (11)}$$

$$\Theta = C_0 \theta + D_0 \quad \text{--- (12)}$$

The solutions of Laplace's equation in cylindrical co-ordinates when the function u is independent of z are called circular harmonics and the number n is called the degree of harmonic. The circular harmonics $U_0(r, \theta)$ and $U_n(r, \theta)$ of degree zero and n are respectively given by

$$U_0(r, \theta) = (A_0 \log r + B_0) (C_0 \theta + D_0) \quad \text{--- (13)}$$

$$U_n(r, \theta) = (A_n r^n + B_n r^{-n}) (C_n \cos n\theta + D_n \sin n\theta) \quad \text{--- (14)}$$

A general single valued solution of Laplace's equation may be obtained by summing up the solution (13) and (14) for all integral values of n and thus we have

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3	10 17 24 31 Sat	7	14 21 28
4	11 18 25 Sun	8	15 22 29

$$u = a_0 \log_0 r + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$+ \sum_{n=1}^{\infty} r^{-n} (c_n \cos n\theta + d_n \sin n\theta) + c_0$$

— (15)

where a_0, a_n, b_n, c_n, d_n and c_0 are arbitrary constants.