Solution of Laplace's equation in spherical polar co-ordinates: spherical harmonics

The Laplace's equation

$$\nabla^2 U = 0$$

in spherical polar co-ordinates (r, θ, ϕ) takes the form

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} = 0 \quad \text{--- (1)}$$

From this equation it is obvious that U is the function of (r, θ, ϕ); therefore by the method of separation of variables its solution may be expressed as

$$U(r, \theta, \phi) = R\Theta\Phi = RS \quad \text{--- (2)}$$

where R is function of r only, Θ is the function of θ only and Φ is the function of ϕ only.

The function

$$S(\theta, \phi) = \Theta\Phi \quad \text{--- (3)}$$

is the function of θ and ϕ and is called the surface harmonic. If ϕ is constant, then S is the function of θ only and is called the zonal surface harmonic.

Substituting $u = RS$ from ② in ① and dividing by RS, we get

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{S\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) - \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} \quad \text{——④}$$

In this equation the left hand side is the function of r only and right hand side is the function of θ and ϕ, therefore each side must be equal to the same constant $n(n+1)$ (say), n being a constant.

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = n(n+1)$$

$$r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} - n(n+1)R = 0 \quad \text{——⑤}$$

$$-\frac{1}{S\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) - \frac{1}{S\sin^2\theta}\frac{\partial^2 S}{\partial\phi^3} = n(n+1)$$

or $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2} + n(n+1)S = 0$$

The solution of equation ⑤ is

$$R = Ar^n + Br^{-n-1} \quad \text{——⑦}$$

If $S = S_n$ is the solution of equation (6)
Then the solution of Laplace's equation in spherical polar co-ordinates is expressed as

$$U = RS = (Ar^n + Br^{-n-1})S_n \quad \text{— (8)}$$

This solution is called the spherical harmonic. The subscript n on S_n signifies that the same value of n must be used in both terms of equation (8)

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