Solution of Laplace's equation in two dimensional cylindrical co-ordinates (r, θ).

The Laplace equation

$$\nabla^2 u = 0$$

in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{—— (1)}$$

If we assume that the function u is independent of co-ordinate z, then Laplace's equation in two dimensional cylindrical co-ordinates takes the form.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{—— (2)}$$

In this equation u is the function of r and θ, therefore by the method of separation of variables u may be written as

$$u(r,\theta) = R(r)\,\Theta(\theta) \quad \text{—— (3)}$$

where R the function of r only and Θ is the function of θ only

Substituting this value of u in eq^n (2) we have

$$\frac{\Theta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

Dividing this eq^n by $\frac{R\Theta}{r^2}$ we get

$$\frac{1}{R}\left[r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}\right]=-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} \quad \text{---} (4)$$

In This equation left hand side is the function of r only; while right hand side is the function of θ only, therefore each side must be equal to the same constant n^2 (say)

i.e $$\frac{1}{R}\left[r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}\right]=n^2$$

or $$r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}-n^2R=0 \quad \text{---} (5)$$

and $$-\frac{1}{\Theta}\frac{\partial^2\Theta}{d\theta^2}=n^2$$

or $$\frac{\partial^2\Theta}{\partial\theta^2}+n^2\Theta=0 \quad \text{---} (6)$$

The solution of equation (5) may be expressed as

$$R=A_n r^n+B_n r^{-n},\ n\neq 0 \quad \text{---} (7)$$

while equation (6) represents simple harmonic motion and its solution may be expressed as

$$\Theta=C_n\cos n\theta+D_n\sin n\theta,\ n\neq 0 \quad \text{---} (8)$$

where A_n, B_n, C_n and D_n are arbitrary constants.

If $n = 0$, the equation (5) and (6) take the form

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} = 0 \quad \text{——— (9)}$$

$$\frac{\partial^2 \Theta}{\partial \theta^2} = 0 \quad \text{——— (10)}$$

The solution of these equation may be expressed as

$$R = A_0 \log_e r + B_0 \quad \text{——— (11)}$$

$$\Theta = C_0 \theta + D_0 \quad \text{——— (12)}$$

The solutions of Laplace's equation in cylindrical co-ordinates when the function U is independent of z are called circular harmonics and the number n is called the degree of harmonic. The circular harmonics $U_0(r, \theta)$ and $U_n(r, \theta)$ of degree zero and n are respectively given by

$$U_0(r, \theta) = (A_0 \log r + B_0)(C_0 \theta + D_0) \quad \text{——— (13)}$$

$$U_n(r, \theta) = (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \quad \text{——— (14)}$$

A general single valued solution of Laplace's equation may be obtained by summing up the solutions (13) and (14) for all integral values of n and thus we have

9
$$u = a_0 \log_e r + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

10
$$+ \sum_{n=1}^{\infty} r^{-n} (c_n \cos n\theta + d_n \sin n\theta) + c_0 \quad \text{--- (15)}$$

11

12 where a_0, a_n, b_n, c_n, d_n and c_0 are arbitrary constants.

13