Derivation of Lorentz Transformation Equation

We observe an event in one reference frame S and characterized its location and time by specifying the co-ordinates x, y, z, t of the event. In a second inertial frame S' the same event is recorded as the space time co-ordinates x', y', z', t'. Since each set of four quantities x', y', z', t' corresponds to a set of x, y, z, t, the two sets must be related by a functional relationship. Hence we must have

$$x' = \Phi_1(x, y, z, t)$$
$$y' = \Phi_2(x, y, z, t)$$
$$z' = \Phi_3(x, y, z, t)$$
$$t' = \Phi_4(x, y, z, t)$$

where $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ are, in general, four different functions. Since S and S' are both inertial frames, a particle moving with a uniform velocity along a straight line relative to an observer in S will also appear to have uniform rectilinear motion relative to an observer in S' according to the first postulate. (Newton's first law of motion). Under these conditions, the functions can be proved to be linear, that is,

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t + a_{15}$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t + a_{25}$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t + a_{35}$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t + a_{45}$$

where the coefficients $a_{11}, a_{12} \ldots a_{45}$ are all constants.

We now choose the origins of time in the two frames in such a way that at $t = t' = 0$, the origins O $(x = y = z = 0)$ and O' $(x' = y' = z' = 0)$ coincide. In order to satisfy this we must have $a_{15} = a_{25} = a_{35} = a_{45} = 0$. If we now substitute $x = y = z = 0$ in the above equations, we get

$$x' = a_{14}t;\ \ y' = a_{24}t;\ \ z' = a_{34}t;\ \ t' = a_{44}t$$

hence

$$x' = \frac{a_{14}}{a_{44}}\,t';\ \ y' = \frac{a_{24}}{a_{44}}\,t';\ \ z' = \frac{a_{34}}{a_{44}}\,t'$$

We can simplify the algebra by choosing the relative velocity of the S and S' frames to be along a common $x - x'$ axis and by keeping corresponding planes parallel. For this, $a_{24} = a_{34} = 0$ therefore, the equations take the following forms:

The x-axis coincides continuously with the x'-axis. This will be so only if for $y = 0, z = 0$ (which characterizes points on the x-axis) it always follows that $y' = 0, z' = 0$ (which characterizes points on the x'-axis). Hence, the transformation formulas for y and z must be of the form:

$$y' = a_{22}y + a_{23}z \qquad \text{and} \qquad z' = a_{32}y + a_{33}z$$

That is, the coefficients a_{21}, a_{24}, a_{31}, a_{34} must be zero. Likewise, the $x - y$ plane (which is characterized by $z = 0$) should transform over to the x'-y' plane (which is characterized by $z' = 0$); similarly, for the $x - z$ and $x' - z'$ planes, $y = 0$ should give $y' = 0$. Hence, it follows that a_{23} and a_{32} are zero so that

$$y' = a_{22}y \qquad \text{and} \qquad z' = a_{33}z$$

Suppose that we have a rod lying along the y-axis, measured by S to be of unit length. According to the S' observer, the rod's length will be, a_{22}(i.e., $y' = a_{22} \times 1$).

Now, suppose that the very same rod is brought to rest along the y' axis of the S'-frame. The primed observer must measure the same length (unity) for this rod when it is at rest in his frame as the unprimed observer measures when the rod is at rest with respect to him; otherwise there would be an asymmetry in the frames. In this case, however, the S-observer would measure the rod's length to be $1/a_{22}$ [i.e., $y = 1/a_{22}y' = 1/a_{22} \times 1$]. Now, because of the reciprocal nature of these length measurements, the first postulate requires that these measurements be identical, for otherwise the frames would not be equivalent physically. Hence, we must have.$a_{22} = 1/a_{22}$or$a_{22} = 1$.The argument is identical in determining that$a_{33} = 1$. Therefore, our two middle transformation equations become

$$y' = y \qquad \text{and} \qquad z' = z$$

There remain transformation equations for $x'$and t', namely,

$$x' = a_{11}x + a_{12}y + a_{13}z+a_{14}t$$

and

$$t' = a_{41}x + a_{42}y + a_{43}z+a_{44}t$$

Let us look first at the t'-equation. For reasons of symmetry, we assume that t' does not depend on $y$and z. Otherwise, clocks placed symmetrically in the y-z plane (such as at $+y, -y$or $+z, -z$) about the x-axis would appear to disagree as observed from S', which would contradict the isotropy of space. Hence,$a_{42} = a_{43} = 0$.

As for the x'-equation, we know that a point having $x' = 0$ appears to move in the direction of the positive x-axis with speed v, so that the statement $x' = 0$ must be identical to the statement $x = vt$. Therefore, we expect $x' = a_{11}(x - vt)$ to be the correct transformation equation. (That is, $x = vt$ always gives $x' = 0$ in this equation.)and $a_{12} = a_{13} = 0$. Hence, $x' = a_{11}x - a_{11}vt = a_{11}x + a_{14}t$. This gives us $a_{14} = -va_{11}$, and our four equations have now been reduced to

$$x' = a_{11}(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = a_{41}x+a_{44}t$$

The observer on S thinks that the origin of S' moves along the x-axis with a velocity$+v$ and the observer on S' will think that that the origin of S moves along x'-axis with a velocity$-v$. The equation of motion of O relative to S' will be$x' = -vt'$. But substituting t in terms of t'at $x = 0$, we get $x' = -\frac{a_{11}}{a_{44}} vt'$). Therefore, $\frac{a_{11}}{a_{44}} = 1$, or, $a_{11} = a_{44} = a\ (say)$. So the transformation equations now take the form

$$x' = a(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = a_{41}x + at$$

Now, we use the principle of the constancy of the velocity of light. Let us assume that at the time $t = 0$ a spherical electromagnetic wave leaves the origin of S, which coincides with the origin of S' at that moment. The wave propagates with a speed c in all directions in each inertial frame. Its progress, then, is described by the equation of sphere whose radius expands with time at a rate c in terms of either the primed or unprimed set of coordinates. That is

$$x^2 + y^2 + z^2 = c^2t^2$$

or,

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

Substituting the transformation equations in the second equation we get

$$a^2(x - vt)^2 + y^2 + z^2 = c^2(a_{41}x + at)^2$$

Rearranging the terms gives us

$$(a^2 - c^2{a_{41}}^2)x^2 + y^2 + z^2 - 2(va^2 + c^2a_{41}a)xt = (c^2{a_{44}}^2 - v^2a^2)t^2$$

Since this equation is valid for all values of x and t it must agree with the equation

$$x^2 + y^2 + z^2 = c^2t^2$$

Thus,

$$a^2 - c^2{a_{41}}^2 = 1$$
$$(va^2 + c^2a_{41}a) = 0$$
$$c^2a^2 - v^2a^2 = c^2$$

From the third equation we get

$$a = \frac{1}{\sqrt{1 - v^2/c^2}}$$

From the second equation

$$a_{41} - \frac{av}{c^2} = -\frac{v}{c^2}\frac{1}{\sqrt{1 - v^2/c^2}} \ (substituting\ the\ value\ of\ a)$$

By substituting the value of constants we get the new transformation equations

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$
$$y' = y$$
$$z' = z$$
$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

which are called the Lorentz transformation equation.

Now, consider the given space-time coordinates of the event to be those observed in S' rather than in$S$, the only change allowed by the relativity principle is the physical one of a change in relative velocity from $v$to$-v$.Solvingfor$x$,y,z and t in terms of the primed coordinates we obtain

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$
$$y = y'$$
$$z = z'$$
$$t = \frac{t' + (v/c^2)x'}{\sqrt{1 - v^2/c^2}}$$

For speeds small compared to c, that is, for $v/c \ll 1$, the Lorentz equations should reduce to the (approximately) correct Galilean transformation equations. This is the case, for when$v/c \ll 1$, the Lorentz transformation Equations reduce to

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

which are the classical Galilean transformation equations.