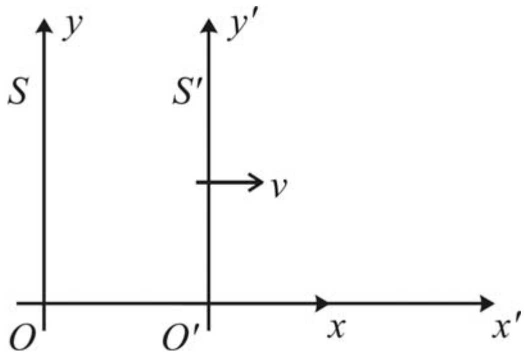
Derivation of the Lorentz transformation formulas:



The two reference frames are *S* and *S'*. *S'* moves along the positive x-direction with a constant speed *v* relative to *S*.

Let the origins *O* and *O'* of the two frames coincide at *t*=*t'*=0. Hence the event (*x*,*t*)=(0,0) should transform as (*x'*,*t'*)=(0,0). This means that the required linear transformation equations don't have a constant term:

$$x' = A \cdot x + B \cdot ct \qquad (1)$$

$$ct' = D \cdot x + E \cdot ct \qquad (2)$$

Our task is to determine the constant factors *A*, *B*, *D* and *E*.

1. Consider a light pulse emitted from *O* at *t*=0 along the positive *x*-axis. The equation of the trajectory of this light pulse is:

$x=c\cdot t$. [light pulse to the right] (3)

The same light pulse – when observed from frame *S'* – originates from *O'* at *t'*=0. Since *S'* measures the velocity of this light pulse to be *c* (just like *S* does) [this is a unique property of light, completely unexplained by classical physics], the equation of the trajectory of the light pulse in *S'* must be:

$x'=c\cdot t'$. [light pulse to the right] (4)

Combining (3) and (4) with (1) and (2) yields:

$A + B = D + E$. (5)

2. Now consider a similar light pulse – emitted from *O* at *t*=0 – but moving along the negative *x*-axis.

The equation of the trajectory of this light pulse is:

$x = -c \cdot t.$ [light pulse to the left] (6)

Again, since *S'* measures the velocity of this light pulse to be *c* (just like *S* does), the equation of the trajectory of the same light pulse in *S'* is:

$x' = -c \cdot t'.$ [light pulse to the left] (7)

Combining (6) and (7) with (1) and (2) yields:

$$A - B = -(D + E). \quad (8)$$

By subtracting and adding eqs. (5) and (8) we get

$$B = D, \quad (9)$$

and

$$A = E, \quad (10)$$

respectively.

The Lorentz transformation equations (1) and (2) are thus simplified to the following symmetrical form:

$$x' = A \cdot x + B \cdot ct \quad (11)$$

$$ct' = B \cdot x + A \cdot ct \quad (12)$$

3. The inverse transformation equations are obtained from eqs. (11) and (12) by solving these two equations for *x* and *ct*:

$$x = \frac{A \cdot x' - B \cdot ct'}{A^2 - B^2} \quad (13)$$

$$ct = \frac{-B \cdot x' + A \cdot ct'}{A^2 - B^2} \quad (14)$$

We can reason in the following way: if *S'* moves to the right with speed *v* relative to *S*, then *S* moves to the left with speed (–*v*) relative to *S'*. Hence the inverse transformation equations (13) and (14) and the transformation equations (11) and (12) should have a 'symmetrical appearance'. Indeed, if the additional condition

$$A^2 - B^2 \equiv 1 \quad (15)$$

is satisfied, eqs. (13) and (14) take a form which is symmetrical to (11) and (12):

$$x = A \cdot x' - B \cdot ct' \quad (16)$$

$$ct = -B \cdot x' + A \cdot ct'. \quad (17)$$

4. Next, let us describe the motion of *O*, the origin of frame *S*, as viewed from *S'*. On the one hand, by substituting $x = 0$ into (11) and (12), we obtain

$$x' = B \cdot ct \,, \qquad \text{[motion of } O\text{]} \qquad (18)$$

$$ct' = A \cdot ct \,, \qquad \text{[motion of } O\text{]} \qquad (19)$$

hence

$$x' = \frac{B}{A} \cdot ct' \,. \qquad \text{[motion of } O\text{]} \qquad (20)$$

On the other hand, as viewed from *S'*, point *O* moves to the left with a speed *v*, and hence its motion is described by

$$x' = -v \cdot t'. \qquad \text{[motion of } O\text{]} \qquad (21)$$

Comparing (20) with (21) yields the following relation between *A* and *B*:

$$B = -A \cdot \frac{v}{c} \qquad (22)$$

5. Finally, solving eqs. (15) and (22) for *A* and *B*, we get

$$A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (23)$$

and

$$B = -\frac{v/c}{\sqrt{1 - \frac{v^2}{c^2}}} \,. \qquad (24)$$

Substituting these values into (11) and (12) we obtain the final form of the Lorentz transformation equations:

$$x' = \frac{x - v \cdot t}{\sqrt{1 - \frac{v^2}{c^2}}} \,, \qquad \text{[Lorentz]} \qquad (25)$$

$$t' = \frac{t - \frac{v}{c^2} \cdot x}{\sqrt{1 - \frac{v^2}{c^2}}} \,. \qquad \text{[Lorentz]} \qquad (26)$$